Wladis, BMCC/CUNY

# Project 1: Limits

**To do this project, all we need to know is:**

1. How to factor polynomial expressions.
2. How to rationalize expressions containing a radical.
3. How to rewrite rational, polynomial and radical definitions in equivalent but different forms.
4. The informal definition of a limit:

**One-sided limits:** The function $f(x)$ has a **right limit** $L$ at $c$ if $f(x)$approaches $L$ as $x$ gets arbitrarily close to $c$ from the right-hand side. We write this as $\lim_{x\to c^+} f(x) = L$. (In other words, as $x$ gets closer and closer to $c$ from the right, $f(x)$ gets closer and closer to $L$. We can similarly define a **left limit**.

**Two-sided limits:** If both the left and right limits are equal, then we say that the **limit** of $f(x)$ at $c$ is $L$, and we write $\lim_{x\to c} f(x) = L$. If $f(x)$ increases without bound as $x$ approaches, $c$, we sometimes write $\lim_{x\to c} f(x) = \infty$. If $f(x)$ decreases without bound as $x$ approaches, $c$, we sometimes write $\lim_{x\to c} f(x) = -\infty$.

**Limits at infinity:** If $f(x)$approaches $L$ as $x$ increases without bound, we say that $\lim_{x\to\infty} f(x) = L$, and if $f(x)$approaches $L$ as $x$ decreases without bound, we say that $\lim_{x\to-\infty} f(x) = L$.

5. How to recognize indeterminate and determinate forms:

**Indeterminate Forms:**

- $\frac{0}{0}$
- $\frac{\pm\infty}{\pm\infty}$
- $\infty - \infty\ (or -\infty + \infty)$
- $0 \cdot \infty$
- $0^0$
- $1^\infty$
- $\infty^0$

**Determinate Forms:** (where $a, b, c$ are arbitrary fixed real numbers, and we assume $a \neq 0, c \neq 1, c > 0$)

- $\frac{a}{0}$
- $\frac{b}{\pm\infty}\ or\ \frac{\pm\infty}{b}$
- $\infty + \infty\ (or -\infty + -\infty)$
- $a \cdot \infty$
- $\pm\infty \cdot \pm\infty$
- $0^\infty$
- $c^\infty$
- $\infty^a\ or\ \infty^{\pm\infty}$

**Our goals on this assignment:**

- To understand the difference between a value that is undefined and a limit expression that is an indeterminate or determinate form.
- To be able to identify the difference between determinate and indeterminate forms when evaluating limit expressions.
- To be able to use tools from algebra to rewrite limit expressions that are indeterminate after direct substitution in order to find the limits of these expressions.

**Example 1:** Find $\lim_{x\to\infty}\frac{x^2-2x+3}{x^3+4}$. If the limit does not exist, explain why.
Here is one example of what a *Prover* and an *Explainer* might do to solve this problem:

$\lim_{x\to\infty}\frac{x^2-2x+3}{x^3+4}=\lim_{x\to\infty}\frac{x^2-2x+3}{x^3+4}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$ *We can multiply both the top and the bottom of the fraction by* $\frac{1}{x^2}$ *without changing the expression (except that now the function is undefined at* $x=0$*, which isn't a problem, since we are interested only in the behavior as* $x\to\infty$*).*

$=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{2x}{x^2}+\frac{3}{x^2}}{\frac{x^3}{x^2}+\frac{4}{x^2}}$ *We multiply the two numerators and the two denominators, making sure to distribute* $\frac{1}{x^2}$ *so that it is multiplied by every single term on both the top and bottom of the fraction.*

$=\lim_{x\to\infty}\frac{1-\frac{2}{x}+\frac{3}{x^2}}{x+\frac{4}{x^2}}$ *We simplify each fraction in the numerator and in the denominator.*

$as\ x\to\infty,\ f(x)\to\frac{1-\frac{2}{\infty}+\frac{3}{\infty^2}}{\infty+\frac{4}{\infty^2}}$ *We plug in the symbol for infinity to try to determine what the behavior of the expression is as* $x$ *increases without bound. We use the arrow sign instead of the equals sign here because the expression containing all the infinity symbols isn't really a number – instead it is a way of describing behavior as* $x$ *gets really large.*

$\to\frac{1-0+0}{\infty+0}=1/\infty$ *We replace* $\frac{2}{\infty}$ *with* $0$ *since as we divide two by larger and larger numbers, the result gets closer and closer to zero. For the same reason, we replace* $\frac{3}{\infty^2}$ *and* $\frac{4}{\infty^2}$ *with* $0$ *also.*

$\to 0$ *We replace* $\frac{1}{\infty}$ *with* $0$ *since as we divide one by larger and larger numbers, the result gets closer and closer to zero.*

$\lim_{x\to\infty}\frac{x^2-2x+3}{x^3+4}=0$ *So the limit of this expression as* $x$ *increases without bound is 0, since* $f(x)$ *gets closer and closer to zero as* $x$ *gets larger and larger.*

**Now you try!**

**1.** Solve the problem from example one in a different way: $\lim_{x\to\infty}\frac{x^2-2x+3}{x^3+4}$. Use the steps in example one above as a model, but this time, multiply the top and bottom of the fraction by $\frac{1}{x^3}$ in the first step.

For each of the problems 2-10, evaluate the limit or conclude that the function tends to $\infty, +\infty, or -\infty$. Use example 1 as a model, and rotate through the rolls of *Prover* and *Explainer*, so that each student plays a different role for each problem. The *Prover* should find the limit step-by-step, and the *Explainer* should explain each step.

**2.** $\lim_{x\to\infty}\frac{2x^2+3x+4}{x^2-2x+3}$

**3.** $\lim_{x\to\infty}\frac{x^4-2x^2+6}{x^2+7}$

**4.** $\lim_{x\to+\infty}\left[\sqrt{a^2+x^2}-x\right]$ where $a$ is an arbitrary fixed real number

**5.** $\lim_{x\to-\infty}\left[\sqrt{a^2+x^2}-x\right]$ where $a$ is an arbitrary fixed real number

**6.** $\lim_{x\to1^+}\frac{x-1}{\sqrt{x^2-1}}$

**7.** $\lim_{x\to1^-}\frac{x-1}{\sqrt{x^2-1}}$

8. $\lim_{x\to+\infty}\frac{x^2+1}{x^{\frac{3}{2}}}$

9. $\lim_{x\to 2^-}\frac{\sqrt{4-x^2}}{\sqrt{6-5x+x^2}}$

10. $\lim_{x\to+\infty}\left[\sqrt{x^2+2x}-x\right]$

For 11-14, the *Prover* should find the functions and should calculate the limits, and the *Explainer* should explain each step in words. As usual, students should rotate through the rolls so that each student plays a different role for each problem.

**11.** Suppose $f(x)\to+\infty$ and $g(x)\to-\infty$ as $x\to+\infty$. Find examples of functions $f$ and $g$ with these properties and such that:
   a. $\lim_{x\to+\infty}[f(x)+g(x)]=+\infty$
   b. $\lim_{x\to+\infty}[f(x)+g(x)]=-\infty$
   c. $\lim_{x\to+\infty}[f(x)+g(x)]=A$, where $A$ is any arbitrary real number that you choose yourself.

**12.** Suppose $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to+\infty$. Find examples of functions $f$ and $g$ with these properties and such that:
   a. $\lim_{x\to+\infty}\left[\frac{f(x)}{g(x)}\right]=+\infty$
   b. $\lim_{x\to+\infty}\left[\frac{f(x)}{g(x)}\right]=-\infty$
   c. $\lim_{x\to+\infty}\left[\frac{f(x)}{g(x)}\right]=A$, where $A$ is any arbitrary real number that you choose yourself.

**13.** Suppose $f(x)\to 0$ and $g(x)\to 0$ as $x\to+\infty$. Find examples of functions $f$ and $g$ with these properties and such that:
   a. $\lim_{x\to+\infty}\left[\frac{f(x)}{g(x)}\right]=+\infty$
   b. $\lim_{x\to+\infty}\left[\frac{f(x)}{g(x)}\right]=-\infty$
   c. $\lim_{x\to+\infty}\left[\frac{f(x)}{g(x)}\right]=A$, where $A$ is any arbitrary real number that you choose yourself.

**14.** Suppose $f(x)\to 0$ and $g(x)\to\pm\infty$ as $x\to+\infty$. Find examples of functions $f$ and $g$ with these properties and such that:
   a. $\lim_{x\to+\infty}[f(x)\cdot g(x)]=+\infty$
   b. $\lim_{x\to+\infty}[f(x)\cdot g(x)]=-\infty$
   c. $\lim_{x\to+\infty}[f(x)\cdot g(x)]=A$, where $A$ is any arbitrary real number that you choose yourself.

d. **Extra Credit:** Suppose $f(x)\to 0$ and $g(x)\to 0$ as $x\to 0^+$. Find examples of functions $f$ and $g$ with these properties and such that:
   a. $\lim_{x\to 0^+} f(x)^{g(x)}=0$
   b. $\lim_{x\to 0^+} f(x)^{g(x)}=1$
   e. $lim_{x\to 0^+} f(x)^{g(x)}=A$, where $A$ is any arbitrary real number except 0 and 1 that you choose yourself.