Solving Equations

What does it mean to solve an equation?

Defn: Solving an equation means finding all the values that can be substituted in for the variable(s) to make the equation true.
This set of values is called the solution set.

Examples:

1) Solve for x: $2x = 6$

$x = 3$

$2(3) = 6$

$6 = 6$ ✓ true

2) Solve for y: $x + y = 1$

$y = 1 - x$

$x + (1 - x) = 1$

$x + (1 + -x) = 1$

$x + 1 + -x = 1$

$\underbrace{x + -x} + 1 = 1$

$0 + 1 = 1$

$1 = 1$ ✓ true

Most techniques for solving equations involve replacing the original equation with an equivalent equation that is simpler.

This usually requires several steps, where the equation from the previous step is replaced with an equivalent equation which is simpler.

Our goal when solving equations:

The equations that we obtain in the final step should have the variable for which we are trying to solve all by itself on one side.

Examples:

1) $x = -2.6$

2) solving for n: $n = \frac{PV}{rt}$

3) $-48 = y$

When we are replacing one equation with an equivalent equation, we often talk colloquially about "moving things around" in the equation or rewriting the equation in a different way.

Why are we interested in solving equations?

Equations can be used to model real world behavior:

Examples:

1) Taxicab fare F in Manhatten, based on the number of miles driven, m

$$F = \$2.50 + \$2m$$

2) Speed S of a falling object dropped from a height of 50 feet at time t

$$S = -16t^2 + 50$$

3) Surface area of a cylinder with radius r and height h

$$S = 2\pi r^2 + 2\pi r h$$

What can we do to an equation without changing the solution set?

Defn: Arrow notation:

$P \to Q$ means "If P is true, then Q is true" (Q follows from P.)

$P \leftrightarrow Q$ means $P \to Q$ and $Q \to P$

1) We can add anything we want to both sides of an equation without changing the solution set.

$$A = B \leftrightarrow A + C = B + C$$

Note: We can always rewrite subtraction as adding a negative number, so we can extend this idea to subtraction also.

2) We can multiply both sides of an equation by anything except zero without changing the solution set.

For $C \neq 0$: $A = B \leftrightarrow AC = BC$

Note: Division can always be rewritten as multiplying by the reciprocal, so we can extend this idea to division also.

Caution: If each side of the equation has multiple terms, we must distribute C to every term.

Example:

$$3x^2 + 2x = 8 - 4x \to 2(3x^2 + 2x) = (8 - 4x)\cdot 2 \to 2\cdot 3x^2 + 2\cdot 2x = 8\cdot 2 - 4x\cdot 2$$

How do we decide what to do to an equation?

- When should we add, subtract, multiply or divide - which operation should we use?
- How do we chose the number or expression to +/−/·/÷?

To answer these questions, we need to decide <u>what we want to move</u> in our equation!

Once we've identified <u>what</u> we want to move:

1) What is the thing we want to move currently <u>doing</u> in the equation? Being Added? Subtracted? Multiplied? Dividing something else?
2) We will want to do the <u>opposite</u> to both sides.

Examples:

1) $\boxed{2x + 2y = 10}$ I want to move the 2y.

The 2y is being <u>added</u>.

So the <u>opposite</u> of adding 2y is <u>subtracting 2y</u>. (or adding $-2y$)

So I <u>subtract</u> 2y from <u>both sides</u>.

$$2x + \underbrace{2y - 2y}_{0} = 10 - 2y$$

$$\Rightarrow 2x + 0 = 10 - 2y$$

$$\Rightarrow \boxed{2x = 10 - 2y}$$

equivalent ⇓ they have <u>the same solution set</u>

Now the 2y has been "moved" to the other side.

2) $\boxed{\frac{n}{3} = 27 + n}$ I want to move the 3.

The 3 is currently dividing the n on the left.

So the <u>opposite</u> of dividing by 3 <u>is multiplying by 3</u>.

So I <u>multiply</u> both sides by 3.

$$\left(\frac{n}{3}\right) \cdot \frac{3}{1} = (27 + n) \cdot 3$$

$$\Rightarrow 1\frac{3n}{3} = 3 \cdot 27 + 3n$$

$$\Rightarrow 1n = 81 + 3n$$

$$\Rightarrow \boxed{n = 81 + 3n}$$

equivalent ⇓ <u>same solution set</u>

Now the 3 is on the other side of the equation.

Cancellation and Opposites

What is an opposite?

What does it mean to "cancel" something out?

It depends on the operation!

When adding and subtracting, "cancelling" something out means getting zero. This is because adding zero to something does not change it.

When multiplying and dividing "cancelling" something out means getting one. This is because multiplying something by one does not change it.

Examples:

1) Solve for x: $6x-4=3x+14$

$$\Rightarrow 6x-4-3x=\underbrace{3x+14-3x}_{=0}$$
$$\Rightarrow 3x-4=14$$
$$\Rightarrow 3x\underbrace{-4+4}_{=0}=14+4$$
$$\Rightarrow \frac{\cancel{3}^{1}x}{\cancel{3}}=\frac{18}{3}$$
$$\Rightarrow 1x=6$$
$$\Rightarrow \boxed{x=6}$$

Check:

$$6(6)-4=3(6)+14$$
$$36-4=18+14$$
$$32=32 \checkmark$$

2) Solve for y: $2(3y-2)+5=8-1(2y-5)$

$$\Rightarrow 6y\underbrace{-4+5}=\underbrace{8+-2y+5}$$
$$\Rightarrow 6y+1=13+-2y$$
$$\Rightarrow 6y+1+2y=13+\underbrace{-2y+2y}_{=0}$$
$$\Rightarrow 8y+1=13$$
$$\Rightarrow 8y\underbrace{+1-1}_{=0}=13-1$$
$$\Rightarrow \frac{\cancel{8}^{1}y}{\cancel{8}}=\frac{12 \div 4=3}{8 \div 4=2}$$
$$\Rightarrow 1y=\frac{3}{2}$$
$$\Rightarrow \boxed{y=\frac{3}{2}}$$

Check:

$$2\left(3\left(\tfrac{3}{2}\right)-2\right)+5=8-\left(2\left(\tfrac{3}{2}\right)-5\right)$$
$$2\left(\tfrac{3}{1}\cdot\tfrac{3}{2}-2\right)+5=8-\left(\tfrac{\cancel{2}^{1}}{1}\cdot\tfrac{3}{\cancel{2}_{1}}-5\right)$$
$$2\left(\tfrac{9}{2}-2\right)+5=8-\left(\tfrac{3}{1}-5\right)$$
$$2\left(\tfrac{9}{2}-\tfrac{2\cdot 2}{1\cdot 2}\right)+5=8-(3-5)$$
$$2\left(\tfrac{9}{2}-\tfrac{4}{2}\right)+5=8-(-2)$$
$$2\left(\tfrac{5}{2}\right)+5=8+2$$
$$\tfrac{\cancel{2}^{1}}{1}\cdot\tfrac{5}{\cancel{2}_{1}}+5=10$$
$$5+5=10 \checkmark$$

3) Solve for a: $6a-4=2(8+3a)$

$\Rightarrow 6a-4=16+6a$

$\Rightarrow \underbrace{6a-4-6a}_{0} = 16+\underbrace{6a-6a}_{0}$

$-4=16 \Rightarrow$ not true!

$\boxed{\text{No solution!}}$ Solution set: $\{\}$

4) Solve for p: $2p+3-8p=3(1-2p)$

$\Rightarrow -6p+3=3-6p$

$\Rightarrow \underbrace{-6p+3+6p}_{0} = 3\underbrace{-6p+6p}_{0}$

$3=3$ ✓ $\rightarrow$ Always true!

Always true, no matter what we plug in for p!

Solution = set of all real numbers

5) Solve for R: $PV = nRT$

$$\Rightarrow \frac{PV}{n} = \frac{\cancel{n}RT}{1\cancel{n}}$$

$$\Rightarrow \frac{PV}{n} = RT$$

$$\Rightarrow \frac{PV}{nT} = \frac{R\cancel{T}}{\cancel{T}}1$$

$$\Rightarrow \boxed{\frac{PV}{nT} = R}$$

$$\Rightarrow \boxed{R = \frac{PV}{nT}}$$

6) Solve for h: $S = 2\pi r^2 + 2\pi r h$

$$\Rightarrow S - 2\pi r^2 = \underbrace{2\pi r^2 + 2\pi r h - 2\pi r^2}_{0}$$

$$\Rightarrow \frac{S - 2\pi r^2}{2\pi r} = \frac{2\pi r h}{1 \; 2\pi r}$$

$$\Rightarrow \frac{S}{2\pi r} - \frac{\cancel{2\pi} r^{\cancel{2}}}{\cancel{2\pi r}} = 1h$$

$$\Rightarrow \boxed{\frac{S}{2\pi r} - r = h}$$

$$\Rightarrow \boxed{h = \frac{S}{2\pi r} - r}$$

The equations we've looked at so far have been linear with respect to the variable of interest.

For single variable equations:

Defn: A linear equation (for the variable x) is an equation that can be put into the form:

$$ax + b = 0$$

for some constants $a + b$ where $a \neq 0$

Examples:

Linear: $2x - 9 = x$

$\Rightarrow 1x - 9 = 0$

$-27 = a$

$1a + 0 = -27$

$\Rightarrow 1a + 27 = 0$

Not Linear: $\sqrt{y} + 6 = 8$

$p^2 + 9 = p - 4$

Defn: A $\underline{\text{quadratic equation}}$ (for the variable x) is an equation that can be put into the form:

$ax^2+bx+c=0$ for some constants $a, b + c$ where $a \neq 0$

Examples:

Quadratic:

$(z+1)(z-9)=0$

$\Rightarrow z^2-9z+z-9=0$

$\Rightarrow z^2-8z-9=0$ ✓

$8=3x^2$

$\Rightarrow -3x^2+8=0$

$\Rightarrow \underline{-3}x^2+\underline{0}x+\underline{8}=0$

not Quadratic:

$\sqrt[3]{3a^2-1}=8-a$

$24p=2^p$

How do we solve quadratic equations?

Example: Solve for x $\underbrace{x^2}-\underbrace{x}=6$
not like terms!

Problem: Since we can't combine terms that contain both xs and x^2s together into a single term, there is no way to use only what we've already used with linear equations to get x by itself on one side in the quadratic case.

We need to notice a neat trick!

If we rewrite this as:

$$x^2-x-6=0$$ (by subtracting 6 from both sides)

Then we notice that we can factor the left side;

$$(x+2)(x-3)=0$$ and the right side is zero.

How does that help us?

Two things can only multiply to get **zero** if one of them **is zero**!

Zero-Factor Property

For any two real numbers, $a + b$: $a \cdot b = 0 \rightarrow a = 0$ or $b = 0$ (or both)

(Or more generally, for any real numbers $a_1, a_2, a_3, \ldots, a_n$:

$$a_1 \cdot a_2 \cdot a_3 \cdots a_n = 0 \rightarrow a_1 = 0 \text{ or } a_2 = 0 \text{ or } a_3 = 0 \text{ or } \ldots$$
$$\ldots \text{ or } a_n = 0 \text{ (or some combination)})$$

Back to our specific example:

$$(x+2)(x-3) = 0 \rightarrow x+2=0 \text{ or } x-3=0$$
$$\quad -2 \;\; -2 \qquad\qquad +3 \;\; +3$$
$$\boxed{x = -2 \text{ or } x = 3}$$

Check!

$(-2)^2 - (-2) - 6 = 0$
$4 + 2 - 6 = 0$
$6 - 6 = 0$ ✓

$(3)^2 - (3) - 6 = 0$
$9 - 3 - 6 = 0$
$6 - 6 = 0$ ✓

Examples:

1) Solve for x: $3x^2 - 17x + 10 = 0$

$$\Rightarrow (3x-2)(x-5) = 0$$

$$\Rightarrow 3x - \cancel{2} = 0 \quad \text{or} \quad x - \cancel{5} = 0$$

$$\quad +\cancel{2} \quad +2 \qquad\qquad +\cancel{5} \quad +5$$

$$\quad 0 \qquad\qquad\qquad\qquad 0$$

$$\frac{\cancel{3}x}{\cancel{3}_1} = \frac{2}{3} \qquad\qquad \boxed{x = 5} \checkmark$$

$$\cancel{1}x = \frac{2}{3}$$

$$\boxed{x = \frac{2}{3}} \checkmark$$

Check: $3\left(\frac{2}{3}\right)^2 - 17\left(\frac{2}{3}\right) + 10 = 0$

$$\Rightarrow 3\left(\frac{4}{9}\right) - \frac{17}{1}\cdot\frac{2}{3} + 10 = 0$$

$$\Rightarrow \frac{\cancel{3}^1}{1}\cdot\frac{4}{\cancel{9}_3} - \frac{34}{3} + 10 = 0$$

$$\Rightarrow \frac{4}{3} - \frac{34}{3} + 10 = 0$$

$$\Rightarrow -\frac{30}{3} + 10 = 0$$

$$\Rightarrow -10 + 10 = 0 \checkmark$$

$3(5)^2 - 17(5) + 10 = 0$

$$3 \cdot 25 - 85 + 10 = 0$$

$$75 - 85 + 10 = 0$$

$$-10 + 10 = 0 \checkmark$$

2) Solve for a: $12a^2 = 2 - 2a$

$-12a^2 \qquad -12a^2$

$\Rightarrow 0 = 2 - 2a - 12a^2$

$\Rightarrow 0 = -12a^2 - 2a + 2$

$\Rightarrow 0 = -(12a^2 + 2a - 2)$

$\Rightarrow \frac{0}{-1} = \frac{-1(3a-1)(4a+2)}{-1}$

$\Rightarrow 0 = (3a-1)(4a+2)$

$\Rightarrow 0 = 3a - 1$ or $0 = 4a + 2$

$+1 \quad +1 \qquad -2 \quad -2$

$\frac{1}{3} = \frac{\cancel{3}a}{\cancel{3}}$ $\qquad$ $\frac{-2}{4} = \frac{\cancel{4}a}{\cancel{4}}$

$\frac{1}{3} = 1a$

$\boxed{\frac{1}{3} = a}$ ✓ or $\boxed{-\frac{1}{2} = a}$ ✓

check!

$12\left(\frac{1}{3}\right)^2 = 2 - 2\left(\frac{1}{3}\right)$

$12\left(\frac{1}{9}\right) = 2 - \frac{2}{1} \cdot \frac{1}{3}$

$\frac{\cancel{12}^4}{1} \cdot \frac{1}{\cancel{9}_3} = 2 - \frac{2}{3}$

$\frac{4}{3} = \frac{2 \cdot 3}{1 \cdot 3} - \frac{2}{3}$

$\frac{4}{3} = \frac{6}{3} - \frac{2}{3}$ ✓

$12\left(-\frac{1}{2}\right)^2 = 2 - 2\left(-\frac{1}{2}\right)$

$12\left(\frac{1}{4}\right) = 2 - \frac{2}{1} \cdot -\frac{1}{2}$

$\frac{\cancel{12}^3}{1} \cdot \frac{1}{\cancel{4}_1} = 2 - \frac{2}{1} \cdot -\frac{1}{2}$

$3 = 2 - {}^-1$

$3 = 2 + 1$ ✓

3) Solve for y: $(2y-1)(y+3)=9$ $\to$ not zero!

$$\Rightarrow 2y^2+6y-y-3=9$$

$$\Rightarrow 2y^2+5y-3=9$$

$$-9 \quad -9$$

$$\Rightarrow 2y^2+5y-12=0$$

$$\Rightarrow (y+4)(2y-3)=0$$

$$\Rightarrow y+4=0 \text{ or } 2y-3=0$$

$$\boxed{y=-4} \checkmark \text{ or } \boxed{y=\tfrac{3}{2}} \checkmark$$

check:

$$(2(-4)-1)((-4)+3)=9$$

$$(-8-1)(-1)=9$$

$$-9\cdot -1=9 \checkmark$$

$$\left(2\left(\tfrac{3}{2}\right)-1\right)\left(\left(\tfrac{3}{2}\right)+3\right)=9$$

$$\left(\tfrac{\cancel{2}^1}{1}\cdot\tfrac{3}{\cancel{2}_1}-1\right)\left(\tfrac{3}{2}+\tfrac{3\cdot 2}{1\cdot 2}\right)=9$$

$$(3-1)\left(\tfrac{3}{2}+\tfrac{6}{2}\right)=9$$

$$2\cdot\tfrac{9}{2}=9$$

$$\tfrac{2}{1}\cdot\tfrac{9}{2}=9 \checkmark$$

4) Solve for p: $12p^4 = 27p^2$

$$12p^4 - 27p^2 = 27p^2 - 27p^2$$

$$\Rightarrow 12p^4 - 27p^2 = 0$$

$$\Rightarrow 3p^2(4p^2 - 9) = 0$$

$$\Rightarrow 3p^2\left((2p)^2 - (3)^2\right) = 0$$

$$\Rightarrow \underline{3p^2}\,\underline{(2p+3)}\,\underline{(2p-3)} = 0$$

$\Rightarrow \frac{\cancel{3}p^2}{\cancel{3}} = \frac{0}{3}$ or $2p + 3 = 0$ or $2p - 3 = 0$

$2p + 3 - 3 = 0 - 3$

$\frac{\cancel{2}p}{\cancel{2}} = \frac{-3}{2}$

$p^2 = 0$

$\boxed{p = 0}$ ✓ or $\boxed{p = -\frac{3}{2}}$ ✓ or $\boxed{p = \frac{3}{2}}$ ✓

Check:

$12(0)^4 \stackrel{?}{=} 27(0)^2$

$12 \cdot 0 = 27 \cdot 0$

$0 = 0$ ✓

$12\left(-\frac{3}{2}\right)^4 = 27\left(-\frac{3}{2}\right)^2$

$\frac{\cancel{12}^{3}}{1} \cdot \frac{81}{\cancel{16}_{4}} = \frac{27}{1} \cdot \frac{9}{4}$

$\frac{243}{4} = \frac{243}{4}$ ✓

$12\left(\frac{3}{2}\right)^4 = 27\left(\frac{3}{2}\right)^2$

$\frac{243}{4} = \frac{243}{4}$ ✓