Exponents (Integer)

Definition: Positive Integer Exponents

If n is a positive integer,

exponent

then $x^n = \underbrace{x \cdot \ldots \cdot x}_{n\text{-many}}$

power

base

Examples:

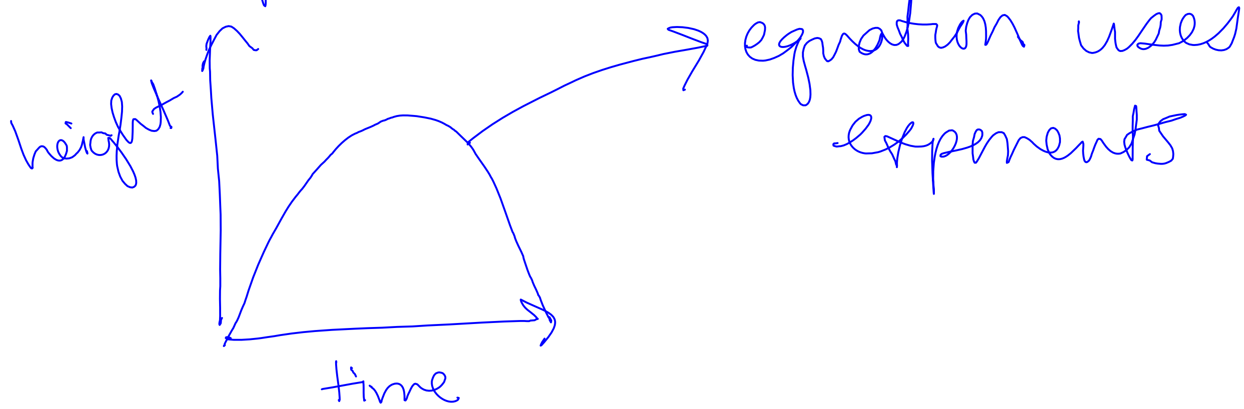
1) $2^3 = 2 \cdot 2 \cdot 2 = 8$

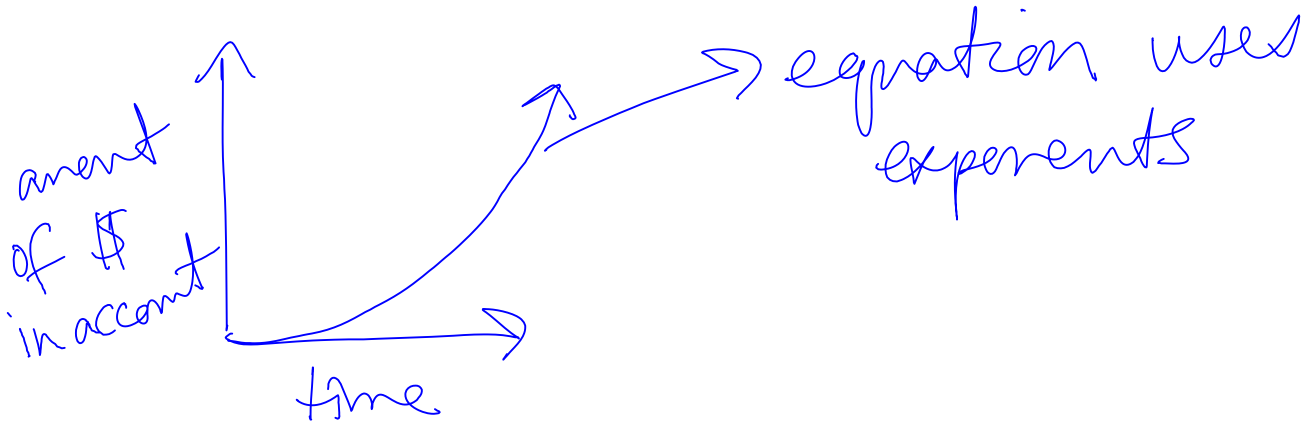
2) $y^1 = y$

Why are we interested in exponents?

They can be used to model real-world behavior.

Examples:





$y = x^2$ $\quad$ $y = e^x$

$$y = \frac{3x^2 - 2x - 5}{(x-2)^3}$$

What do we want to be able to do with exponents?

We want to be able to take expressions or equations that contain exponents and to rewrite them so that we have an equivalent expression or equation that looks different from the one we started with.

Example:

1) Simplify expressions or equations that contain exponents

2) Writing expressions or equations in a different form sometimes allow use to apply a particular technique that we could not use on the original one.

Exponent Patterns

Example: Multiplying Powers

$$x^2 \cdot x^5 = \underbrace{(x \cdot x)}_{\text{2-many}}\underbrace{(x \cdot x \cdot x \cdot x \cdot x)}_{\text{5-many}}$$

$$= x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$$

$$= x^7 \qquad 7 = 2+5$$

Add Exponents?

Exponent Pattern: Multiplying Powers with the same base

$$a^n a^m = \underbrace{(a \cdot \cdots \cdot a)}_{n\text{-many}}\underbrace{(a \cdot \cdots \cdot a)}_{m\text{-many}}$$

$$= \underbrace{\overbrace{a \cdot \cdots \cdot a}^{} \cdot \overbrace{a \cdot \cdots \cdot a}^{}}_{(n+m)\text{-many}}$$

$$= a^{n+m}$$

Add the Exponents!

More Exponent Patterns

$$(y^2)^3 = (y \cdot y)^3$$
$$= (y \cdot y)(y \cdot y)(y \cdot y)$$
$$= y \cdot y \cdot y \cdot y \cdot y \cdot y$$
$$= y^6$$

$$(ab)^4 = (ab)(ab)(ab)(ab)$$
$$= abababab$$
$$= \underbrace{a \cdot a \cdot a \cdot a}_{4 \text{ times}} \cdot \underbrace{b \cdot b \cdot b \cdot b}_{4 \text{ times}}$$
$$= a^4 b^4$$

$$\frac{p^5}{p^3} = \frac{p \cdot p \cdot \cancel{p} \cdot \cancel{p} \cdot \cancel{p}}{\cancel{p} \cdot \cancel{p} \cdot \cancel{p}} = \frac{p \cdot p}{1} = p \cdot p = p^2$$

$5 - 3 = 2$

$$\left(\frac{x}{y}\right)^3 = \underbrace{\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)} = \frac{x \cdot x \cdot x}{y \cdot y \cdot y} = \frac{x^3}{y^3}$$

Zero Exponents

$$\frac{a^n}{a^n} = a^{n-n} = a^0$$ according to the pattern for positive integer exponents

$$\frac{a^n}{a^n} = 1$$

$$\Rightarrow \boxed{a^0 = 1}$$

Definition: Zero exponents:

For $a \neq 0$, $a^0 = 1$

Checking Exponent Patterns for Zero Exponents

1) $a^n a^m = a^{n+m}$ ✓

i) $a^0 a^m = 1 \cdot a^m = a^m$ ✓ $0+m=m$

i) $a^n a^0 = a^n \cdot 1 = a^n$ ✓ $n+0=n$

ii) $a^0 a^0 = 1 \cdot 1 = 1 = a^0$ ✓ $0+0=0$

Negative Integer Exponents

1) $a^n a^m = a^{n+m}$

$$a^n a^{-m} = a^{n+-m}$$
$$= a^{n-m}$$
$$= \frac{a^n}{a^m}$$

$$a^n a^{-m} = \frac{a^n}{a^m} = a^n \cdot \frac{1}{a^m}$$

$$\frac{\cancel{a^n} a^{-m}}{\cancel{a^n}} = \frac{\cancel{a^n} \cdot \frac{1}{a^m}}{\cancel{a^n}}$$

$$\boxed{a^{-m} = \frac{1}{a^m}}$$

Definition: Negative Integer Exponents

For $a \neq 0$, m a positive integer

$$a^{-m} = \frac{1}{a^m}$$

1) $a^n a^m = a^{n+m}$

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ii) $a^{-n} a^m = \frac{1}{a^n} \cdot a^m$

$= \frac{a^m}{a^n}$

$= a^{m-n}$

$= a^{m+-n}$

$= a^{-n+m}$ ✓ $-n+m$

iii) $a^{-n} a^{-m} = \frac{1}{a^n} \cdot \frac{1}{a^m}$

$= \frac{1}{a^n a^m}$

$= \frac{1}{a^{(n+m)}}$

$= a^{-(n+m)}$

$= a^{-n+-m}$ ✓ $-n+-m$

Negative signs in the base or in front of the power?

Example:

$\rightarrow (-3)^4 = (-3)(-3)(-3)(-3) = 81$

$\rightarrow \underbrace{-3^4} = \underbrace{-(3^4)}$

$-3^4 = -(3^4) = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$

$\boxed{\text{Definition: } a \neq 0,\ n \text{ is an integer} \quad -a^n = -(a^n)}$

Examples:

Simplify the expression containing exponents:

1) $(3x^2)(4x^{-8}) = 3 \cdot 4 \cdot x^2 \cdot x^{-8}$

$= \boxed{12x^{-6}} = 12 \cdot \frac{1}{x^6} = \boxed{\frac{12}{x^6}}$

Write the answer with positive exponents only.

$= 3 \cdot 4 \cdot x^2 \cdot x^{-8}$

$= 12 \cdot x^2 \cdot x^{-8}$

$= \frac{12}{1} \cdot \frac{x^2}{1} \cdot \frac{1}{x^8}$

$= \frac{12x^2}{x^8} = \frac{12 \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x} = \frac{12}{x \cdot x \cdot x \cdot x \cdot x \cdot x}$

$= \boxed{\frac{12}{x^6}}$

Simplify so that the expression contains only positive exponents:

2) $$\frac{\overset{\div 2}{4}ab^{-1}}{\underset{\div 2}{2}a^2b^2} = \frac{2ab^{-1}}{1a^2b^2} = \frac{2ab^{-1}}{a^2b^2}$$

$$= \frac{2a^1b^{-1}}{a^2b^2}$$

$1-2=-1$

$-1-2=-1+-2$

$=-3$

$$= 2a^{(1-2)}b^{(-1-2)}$$

$$= 2a^{-1}b^{-3}$$

$$= \frac{2}{1}\cdot\frac{1}{a^1}\cdot\frac{1}{b^3}$$

$$= \frac{2}{a^1b^3}$$

$$= \boxed{\frac{2}{ab^3}}$$

3) $(2x^2y^{-3})^4(3x^{-5}y)$

$= 2^4 \cdot (x^2)^4 \cdot (y^{-3})^4 \cdot (3x^{-5}y)$

$= 16x^8y^{-12} \cdot (3x^{-5}y)$

$= 16 \cdot 3 \cdot \underbrace{x^8 \cdot x^{-5}} \cdot \underbrace{y^{-12} \cdot y^1}$

$8 + -5 = 3$

$-12 + 1 = -11$

$= 48x^3y^{-11}$

$= 48x^3 \cdot \dfrac{1}{y^{11}}$

$= \boxed{\dfrac{48x^3}{y^{11}}}$

4) $\dfrac{7a^{-4}b}{28(ab^{-2})^{-3}} = \dfrac{1a^{-4}b^{1}}{4a^{-3}(b^{-2})^{-3}}$ $\quad -2\cdot-3=6$

$= \dfrac{a^{-4}b^{1}}{4a^{-3}b^{6}}$ $\quad -4--3=-4+3=-1$ $\quad 1-6=-5$

$= \dfrac{1}{4}\cdot a^{-1}\cdot b^{-5}$

$= \dfrac{1}{4}\cdot\dfrac{1}{a^{1}}\cdot\dfrac{1}{b^{5}}$

$= \boxed{\dfrac{1}{4ab^{5}}}$

5) $\left(\dfrac{15p^{2}q^{-1}}{5p^{-2}q}\right)^{-1} = \left(\dfrac{3p^{2}q^{-1}}{p^{-2}q^{1}}\right)^{-1}$ $\quad 2--2=2+2=4$ $\quad -1-1=-1+-1=-2$

$= (3p^{4}q^{-2})^{-1}$

$= \dfrac{1}{3p^{4}q^{-2}} = \dfrac{1}{3p^{4}}\cdot\dfrac{1}{q^{-2}}$ $\quad 1\div\dfrac{1}{q^{2}}=1\cdot\dfrac{q^{2}}{1}$ $= q^{2}$

$= \dfrac{1}{3p^{4}}\cdot\dfrac{1}{\frac{1}{q^{2}}}$

$= \dfrac{1}{3p^{4}}\cdot\left(1\div\dfrac{1}{q^{2}}\right)$

$= \dfrac{1}{3p^{4}}\cdot\dfrac{q^{2}}{1}$

$= \boxed{\dfrac{q^{2}}{3p^{4}}}$

Another Pattern:

$$\underbrace{\frac{1}{a^{-n}} = 1 \div a^{-n} = 1 \div \frac{1}{a^n} = 1 \cdot \frac{a^n}{1} = a^n}$$

$$\boxed{\frac{1}{a^{-n}} = a^n \text{ or } \frac{a^n}{1}}$$

Exponent Patterns $(a, b \neq 0)$

$a^n a^m = a^{n+m}$

$\frac{a^n}{a^m} = a^{n-m}$

$(a^n)^m = a^{nm}$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$(ab)^n = a^n b^n$