Project 7: "Minigoals" Towards Simplifying and Rewriting Expressions

The distributive property and "like terms"

You have probably learned in previous classes about "adding like terms"—but one problem with the way that we often think about "like terms" is that it isn't always clear what is meant by "like". Really, combining "like terms" is really just applying the **<u>distributive property</u>** in reverse:

$$ac + bc = (a + b)c$$

Or, more generally (for any number of terms, where $n$just denotes the number of terms) we could write:

$$a_1c + \cdots + a_nc = (a_1 + \cdots + a_n)c$$

Any time we have a sequence of terms, each of which is a product (two things being multiplied together), and the <u>second factor in each product is the SAME</u>, then we can used the distributive property to rewrite the expression.

So, now let's work on practicing when we can apply the distributive property.

<u>Examples:</u>

Which of these have the form of the left side of one of the distributive properties above?

For each one, CIRCLE c, and identify a, b, c or $a_1, \ldots, a_n, c$:

$5x - 3x$	The distributive property can be applied if we rewrite the subtraction as adding a negative <u>first</u>: $5x - 3x = 5x + -3x$ So $5\boxed{x} + -3\boxed{x}$ And $a = 5, b = -3, c = x$ Or we could write $a_1 = 5, a_2 = -3, c = x$
$5x - 3y$	The distributive property can NOT be applied here because these two terms don't have anything in common. I cannot find anything of the form $ac + bc$, even if I rewrite the subtraction as adding a negative first.
$8p^{-2}q^2 - 12p^{-2}q^2$	The distributive property can be applied if we rewrite the subtraction as adding a negative <u>first</u>: $8p^{-2}q^2 - 12p^{-2}q^2 = 8p^{-2}q^2 + -12p^{-2}q^2$ So $8\boxed{p^{-2}q^2} + -12\boxed{p^{-2}q^2}$ And $a = 8, b = -12, c = p^{-2}q^2$ Or we could write $a_1 = 8, a_2 = -12, c = p^{-2}q^2$
$8\sqrt{5} - 5\sqrt{5} + 2\sqrt{5}$	The distributive property can be applied if we rewrite the subtraction as adding a negative <u>first</u>: $8\sqrt{5} - 5\sqrt{5} + 2\sqrt{5} = 8\sqrt{5} + -5\sqrt{5} + 2\sqrt{5}$ So $8\boxed{\sqrt{5}} + -5\boxed{\sqrt{5}} + 2\boxed{\sqrt{5}}$ And $a_1 = 8, a_2 = -5, a_3 = 2, c = \sqrt{5}$ Note: here we have to use the formula with $a_1, \ldots, a_n$ in it, because here we have <u>more than two</u> terms.
$2\sqrt{3} + 3\sqrt{2}$	The distributive property can NOT be applied here because these two terms don't have anything in common. I cannot find anything of the form $ac + bc$, even if I use the commutative property $ab = ba$ to reorder the factors in each term first. Note: 2 is NOT the same as $\sqrt{2}$, and $\sqrt{3}$ is NOT the same as $\sqrt{2}$.

$5ab + 4b$	The distributive property can be applied if we think about how to group all the factors in the first term, and then group the $5a$ together and treat it as one unit: $5ab + 4b = (5a)b + 4b$ So $(5a)\boxed{b} + 4\boxed{b}$ Here I have a's and b's in my expression and in the distributive property identity, so I am going to change the letters in the first distributive property like this: $ac + bc = (a + b)c \quad \rightarrow \quad pr + qr = (p + q)r$ Then $p = 5a, q = 4, r = b$ Or we could write $a_1 = 5a, a_2 = 4, c = b$
$3x(x - 1) + 2(x - 1)$	The distributive property can be applied: So $3x\boxed{(x - 1)} + 2\boxed{(x - 1)}$ And $a = 3x, b = 2, c = (x - 1)$ Or we could write $a_1 = 3x, a_2 = 2, c = (x - 1)$
$5x - 3x^2$	The distributive property can be applied if we 1) rewrite the subtraction as adding a negative, and think about how to group all the factors in the second term, after remembering that x^2 can be replaced with $x \cdot x$: $5x - 3x^2 = 5x + -3 \cdot x \cdot x = 5x + (-3x) \cdot x$ So $5\boxed{x} + (-3x)\boxed{x}$ And $a = 5, b = -3x, c = x$ Or we could write $a_1 = 5, a_2 = -3x, c = x$

Do we always want to use the distribute property in examples like this?

For each of these examples, we could see how we can apply the distributive property in many cases to combine terms, even in many cases that you probably didn't previously think of as "like terms". In more complex cases like these, we may or may not want to apply the distributive property in reverse—it will depend on our goals with the particular expression. But it is important to understand that the distributive property CAN be used in all of these cases.

<u>Now you try!</u>

Which of these have the form of the left side of one of the distributive properties above?

For each one:

a) State whether or not one of the distributive properties can be applied;
b) If it can, explain which identities, if any, need to be applied first; if it can NOT, explain WHY not;
c) CIRCLE c; and
d) identify a, b, c (or $a_1, \ldots, a_n, c$, depending on the distributive property that you are using) from one of the two given distributive properties:

1. **Two terms:** $\boldsymbol{ac + bc = (a + b)c}$
2. **Any number of terms:** $\boldsymbol{a_1c + \cdots + a_nc = (a_1 + \cdots + a_n)c}$

Use the examples above as a model.

1. $2a - 3a$	

2. $2a - 3b$	
3. $2a^2 - 3a$	
4. $4x^{-3}y^2 + 5x^{-3}y^2$	
5. $3\sqrt{7} - \sqrt{7}$	
6. $4\sqrt{3} - 3\sqrt{2} + \sqrt{3}$	
7. $2xy - 3y$	
8. $2x(2x + 1) + 3(2x + 1)$	

Rewriting them with a specific goal in mind

Typically when we rewrite expressions by replacing them with equivalent ones, it is with a specific goal in mind. We may want to replace them with the simplest possible equivalent expression, or we may want to replace them with an expression that has a particular form that allows us to see relationships (for example, equations with a particular form can be more easily graphed or solved).

When we have a goal in mind for rewriting an expression (or equation) by replacing it with a sequence of equivalent ones, we then have to put together many of the things that we have already been doing so that we apply identities to the expressions one after another. This requires several steps:

1) We have to identify our end goal (e.g. Why are we rewriting the expression/equation? What does it need to look like when we are done?).
2) We have to select particular identities that could be applied to the current expression (either in its current form, or with a little rewriting), and we need to consider each of these possible identities as a way of replacing the expression with an equivalent one that gets us closer to our goal. If the rewriting is somewhat complex, this may require us to set some smaller goals that along the way to meet our larger goal.
3) Each time we apply an identity to the expression, we ask ourselves if this brought us closer to our goal or not.
    a. If not, we may need to put that work aside and to start over by applying a different identity to the original equation.
    b. If it has gotten us closer, then we need to think about what identity we should apply to this new equivalent function in order to get even closer to our goal.
4) We keep repeating this process until we reach our goal. As problems grow more complex, we might have to apply many, many identities sequentially to get to our final result.

Let's just jump in and start trying to simplify expressions by using the identities we have been using so far. <u>To keep things simple, we will only need to use the identities on the list below as we work to simplify the next set of problems</u>.

- $x = 1x$
- $a - b = a + -b$
- $a + b = b + a$
- $(a + b) + c = a + b + c$
- $a + (b + c) = a + b + c$
- $ab = ba$
- $(ab)c = abc$
- $a(bc) = abc$
- $a(b + c) = ab + ac$
- $(a + b)c = ac + bc$
- $x^n = \underbrace{x \cdot \ldots \cdot x}_{n-many\ times}$
- $\frac{x}{x} = 1 \quad (whenever\ x \neq 0)$
- $\frac{a}{b} = a \cdot \frac{1}{b} \quad (whenever\ b \neq 0)$
- $\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{d} \quad (whenever\ c, d \neq 0)$
- $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \quad (whenever\ c \neq 0)$
- $\sqrt{ab} = \sqrt{a}\sqrt{b}$
- $\sqrt{x^2} = x \quad (whenever\ x \geq 0)$

Examples:

A) Replace the following expression with the simplest equivalent expression that you can find:

$$(6x^2+5)-(2x^2+6)$$

If our goal is to simplify this, we will likely want to replace it with an equivalent expression without the parentheses so that we can in a future step try to combine what is in each set of parentheses with each other. If that is possible, it will very likely make the expression simpler. **So, our first mini-goal is: Replace the expression with an equivalent one that does not have parentheses.**

If the expression was of this form: $(a+b)+c$, then we could use the identity $(a+b)+c=a+b+c$ to replaced this expression with an equivalent one that does not have the first set of parentheses. But we need to rewrite all of the subtraction in this expression as addition of a negative first. **So this leads us to another mini-goal, which must come <u>BEFORE</u> our goal of replacing the expression with an equivalent one that does not have parentheses: Rewrite all subtraction in the expression as addition of a negative.**

Let's put these two goals in the order that we need to do them, and list the identities needed to accomplish each goal:

Mini-goal	Identities needed
1) Rewrite all subtraction in the expression as addition of a negative.	$a-b=a+-b$
2) Replace the expression with an equivalent one that does not have parentheses. We will start with the first set of parentheses and then focus on the second set.	$(a+b)+c=a+b+c$

Now let's try to do these two steps, and see what happens:

Step 1: Rewrite the subtraction in the expression as addition of a negative, using the identity $a-b=a+-b$

$$a=(6x^2+5),\ b=(2x^2+6)\ \rightarrow\ a-b=a+-b\ \rightarrow\ \overbrace{(\quad)}^{a}-\overbrace{(\quad)}^{b}=\overbrace{(\quad)}^{a}+-\overbrace{(\quad)}^{b}$$

$$\rightarrow\ \overbrace{(6x^2+5)}^{a}-\overbrace{(2x^2+6)}^{b}=\overbrace{(6x^2+5)}^{a}+-\overbrace{(2x^2+6)}^{b}$$

So $(6x^2+5)-(2x^2+6)=(6x^2+5)+-(2x^2+6)$

Step 2: Replace the expression with an equivalent one that does not have (the first set of) parentheses, using the identity $(a+b)+c=a+b+c$.

Focusing on the first set of parentheses:

$$a=6x^2,\ \ b=5,\ \ c=-(2x^2+6)\ \rightarrow(a+b)+c=a+b+c$$

$$\rightarrow\ \left(\overbrace{(\quad)}^{a}+\overbrace{(\quad)}^{b}\right)+\overbrace{(\quad)}^{c}=\overbrace{(\quad)}^{a}+\overbrace{(\quad)}^{b}+\overbrace{(\quad)}^{c}$$

$$\rightarrow\ \left(\overbrace{(6x^2)}^{a}+\overbrace{(5)}^{b}\right)+\overbrace{\left(-(2x^2+6)\right)}^{c}=\overbrace{(6x^2)}^{a}+\overbrace{(5)}^{b}+\overbrace{\left(-(2x^2+6)\right)}^{c}$$

So $(6x^2+5)+-(2x^2+6)=6x^2+5+-(2x^2+6)$

Now we need to focus on replacing the expression with an equivalent one that does not have the <u>second</u> set of parentheses. The sub-expression $-(2x^2+6)$ almost looks like the left side of the identity $a(b+c)=ab+ac$, except that the variable $\boldsymbol{a}$ can't stand in for a negative sign alone—the negative sign has to belong to a number or an expression. So now we form some new mini-goals, on our way to the goal of rewriting the expression without parentheses:

Mini-goal: Replace $-(\boldsymbol{2x^2+6})$ with an equivalent expression of the form $\boldsymbol{a(b+c)}$.

We can do this if we notice that we can apply the identity $x=1x$ to the expression $(2x^2+6)$. So, let's do that:

First let's rewrite the identity $x = 1x$ with another variable to avoid confusion: $q = 1q$.

$$q = (2x^2 + 6), \rightarrow \quad q = 1q \quad \rightarrow \quad \overbrace{(\quad)}^{q} = 1\overbrace{(\quad)}^{q} \quad \rightarrow \quad \overbrace{(2x^2 + 6)}^{q} = 1\,\overbrace{(2x^2 + 6)}^{q}$$

So $6x^2 + 5 + -\boxed{(2x^2 + 6)} = 6x^2 + 5 + -\boxed{1(2x^2 + 6)}$

Now, back to our first mini goal:

Mini-goal: Replace the expression with an equivalent expression that does not have the second set of parentheses.

We see that we can now use the identity $a(b + c) = ab + ac$ to replace the second half of the expression with an equivalent sub-expression that does not have the parentheses:

$$a = -1,\ b = 2x^2,\ c = 6 \ \rightarrow \ a(b + c) = ab + ac \ \rightarrow \ \overbrace{(\quad)}^{a}\left(\overbrace{(\quad)}^{b} + \overbrace{(\quad)}^{c}\right) = \overbrace{(\quad)}^{a}\overbrace{(\quad)}^{b} + \overbrace{(\quad)}^{a}\overbrace{(\quad)}^{c}$$

$$\rightarrow \ \overbrace{(-1)}^{a}\left(\overbrace{(2x^2)}^{b} + \overbrace{(6)}^{c}\right) = \overbrace{(-1)}^{a}\,\overbrace{(2x^2)}^{b} + \overbrace{(-1)}^{a}\,\overbrace{(6)}^{c}$$

So $6x^2 + 5 + \boxed{-1(2x^2 + 6)} = 6x^2 + 5 + \boxed{(-1)(2x^2) + (-1)(6)}$

So now we have replaced the original expression with this equivalent but simpler expression:

$6x^2 + 5 + (-1)(2x^2) + (-1)(6)$

How can we replace this expression with another equivalent expression that is even simpler? The most obvious step is for us to perform the multiplication in the last few terms:

$6x^2 + 5 + (-1)(2x^2) + \boxed{(-1)(6)} = 6x^2 + 5 + (-1)(2x^2) + \boxed{-6}$ because $(-1)(6) = -6$

$6x^2 + 5 + \boxed{(-1)(2x^2)} + -6 = 6x^2 + 5 + \boxed{-2x^2} + -6$ because $(-1)(2x^2) = -1 \cdot 2 \cdot x^2 = -2x^2$

So now we have replaced the original expression with this equivalent expression: $6x^2 + 5 + -2x^2 + -6$

Can we replace this with another expression that is even simpler? It would be good for us to look to see if we can combine any of these terms together. For example, if 5 and -6 were next to each other, we could add them. Similarly, if $6x^2$ and $-2x^2$ were next to each other, we could use the identity $(a + b)c = ac + bc$ to combine them. So, our next mini-goal is:

Mini-goal: Use the properties of addition to reorder the terms so that "like" terms are next to each other.

I can use the identity $a + b = b + a$ to do this:

$$a = 5,\ b = -2x^2 \ \rightarrow \ a + b = b + a \ \rightarrow \ \overbrace{(\quad)}^{a} + \overbrace{(\quad)}^{b} = \overbrace{(\quad)}^{b} + \overbrace{(\quad)}^{a} \ \rightarrow \ \overbrace{(5)}^{a} + \overbrace{(-2x^2)}^{b} = \overbrace{(-2x^2)}^{b} + \overbrace{(5)}^{a}$$

So $6x^2 + \boxed{5 + -2x^2} + -6 = 6x^2 + \boxed{-2x^2 + 5} + -6$

Now we can simply add the final two like terms:

$6x^2 + -2x^2 + \boxed{5 + -6} = 6x^2 + -2x^2 + \boxed{-1}$ because $5 + -6 = -1$

And now our final mini-goal is:

Mini-goal: Used the identity $(a + b)c = ac + bc$ to combine the first two terms.

We noted that the first two terms have the form of the <u>right</u> side of the identity.

$$a = 6,\ b = -2,\ c = x^2 \ \rightarrow \ (a + b)c = ac + bc \ \rightarrow \ \left(\overbrace{(\quad)}^{a} + \overbrace{(\quad)}^{b}\right)\overbrace{(\quad)}^{c} = \overbrace{(\quad)}^{a}\overbrace{(\quad)}^{c} + \overbrace{(\quad)}^{b}\overbrace{(\quad)}^{c}$$

$$\rightarrow \ \left(\overbrace{(6)}^{a} + \overbrace{(-2)}^{b}\right)\overbrace{(x^2)}^{c} = \overbrace{(6)}^{a}\,\overbrace{(x^2)}^{c} + \overbrace{(-2)}^{b}\,\overbrace{(x^2)}^{c}$$

So, $\boxed{6x^2 + -2x^2} + -1 = \boxed{(6 + -2)x^2} + -1$ because $(6 + -2)x^2 = 6x^2 + -2x^2$

And finally, we can add the 6 and the -2 inside the parentheses, so we go ahead and do that:

$\boxed{(6+-2)}x^2+-1=\boxed{4}x^2+-1$ *because* $6+-2=4$

So now we have replaced the original expression with the equivalent expression $4x^2+-1$
This is about as simple as it gets, although, we could do one final step and rewrite the addition of a negative as subtraction, if we want:

$$a=4x^2, b=1 \;\rightarrow\; a-b=a+-b \;\rightarrow\; \overbrace{(\quad)}^{a}-\overbrace{(\quad)}^{b}=\overbrace{(\quad)}^{a}+-\overbrace{(\quad)}^{b} \rightarrow \overbrace{(4x^2)}^{a}-\overbrace{(1)}^{b}=\overbrace{(4x^2)}^{a}+-\overbrace{(1)}^{b}$$

So, $4x^2+-1=4x^2-1$.

This is definitely as simple as we could make this expression, so our final expression is $4x^2-1$ *and:*

$$(6x^2+5)-(2x^2+6)=\boxed{4x^2-1}$$

<u>Now you try!</u>

We are going to start trying to replace expressions with a sequence of equivalent expressions with a specific goal in mind. However, first you are going to start by working on problems where you will be given several minigoals in order. You should try to complete each minigoal, and that will help lead you to the overall goal. <u>Be sure to do the minigoals **in order**, and be sure to **work sequentially, applying identities to the results of the previous step**, NOT to the original expression. In many casts, you will **first have to use several identities** to rewrite the expression **before** you can accomplish the minigoal.</u> Use the previous example as a guide to help you work through these problems.

1. Replace the following expression with the simplest equivalent expression that you can find:

$$(2x+3)-(3x+1)$$

Minigoal 1: Apply the identity $a(b+c)=ab+ac$ in order to replace the expression with an equivalent one that does not have parentheses.

Minigoal 2: Apply the identity $a_1c+\cdots a_nc=(a_1+\cdots+a_n)c$ in order to combine "like" terms.

2. Replace the following expression with the simplest equivalent expression that you can find:

$$(2x+3)(3x+1)$$

Minigoal 1: Apply the identity $a(b+c)=ab+ac$ in order to replace the expression with an equivalent one that has fewer parentheses.

Minigoal 2: Apply the identity $(a+b)c=ac+bc$ in order to replace the expression with an equivalent one that does not have parentheses.

Minigoal 3: Apply the identity $a_1c+\cdots+a_nc=(a_1+\cdots+a_n)c$ in order to combine "like" terms.

3. Replace the following expression with the simplest equivalent expression that you can find:

$$3x^2 - 2x + 5x^2 - 5$$

Minigoal 1: Apply the identity $a_1c + \cdots + a_nc = (a_1 + \cdots + a_n)c$ in order to combine "like" terms.

4. Replace the following expression with the simplest equivalent expression that you can find:

$$\sqrt{2xy^3} \cdot \sqrt{2xy} \qquad (x, y \geq 0)$$

Minigoal 1: Apply the identity $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ to replace the expression with an equivalent one in which the whole expression is under a single radical.

Minigoal 2: Use the identity $a \cdot a = a^2$ as many times as needed to rewrite all possible squares (including numbers and variables) as squares.

Minigoal 3: Use the identity $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ to replace this with an equivalent expression in which each square is under its own radical.

Minigoal 4: Use the identity $\sqrt{a^2} = a$ as many times as needed to replace the expression with an equivalent one that has as few radicals as possible.

5. Replace the following expression with the simplest equivalent expression that you can find:

$$\frac{6x^2-3x}{3x} \quad (x \neq 0)$$

Minigoal 1: Use the identity $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ to replace this expression with an equivalent one that contains two fractions, so that you can focus on one fraction at a time.

Minigoal 2: Use the identity $\frac{ab}{c} = \frac{a}{b} \cdot c$ to replace the expression with an equivalent one in which something of the form $\frac{p}{p}$ is isolated by itself.

6. Replace the following expression with the simplest equivalent expression that you can find:

$$\frac{2x^2y^3}{x^4}$$

Minigoal 1: Use the identities $ab = ba$ and $x^n = \underbrace{x \cdot \cdots \cdot x}_{n-many\ times}$ to replace the expression with an equivalent one in which all products of x are lined up in the top and bottom of the fraction, so that we can try to isolate these later in future steps.

Minigoal 2: Use the identity $\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{c}$ to replace the expression with an equivalent one in which something of the form $\frac{p}{p}$ is isolated by itself.

7. Replace the following expression with an equivalent one that has the form $ax^2 + bx + c$, where a, b, c are real numbers:

$$2x(3x - 6) + 5$$

Minigoal 1: Apply the identity $a(b + c) = ab + ac$ in order to replace the expression with an equivalent one that does not have parentheses.

Minigoal 2: Apply the identity $a_1c + \cdots + a_nc = (a_1 + \cdots + a_n)c$ in order to combine "like" terms.