Project 3: Using Identities to Rewrite Expressions

In algebra, equations that describe properties or patterns are often called *identities*. **Identities describe an expression can be replaced with an <u>equal or equivalent</u> expression that has a different form.**

So, for example, consider this identity: $a+b=b+a$. This means that for any two things that we are adding together (numbers, simple expressions, more complex expressions—**anything that the order of operations allows us to group together**), we can always switch the order and the result will still be equal to the original expression.

<u>The actual letters a and b aren't important</u>. We could also write this identity as $x+y=y+x$ or as $p+q=q+p$.

Example:

A) Let's rewrite $a+b=b+a$ using the letter p for a and the letter q for b:

$a=p, \quad b=q$

$a+b=b+a$

$\overbrace{(\quad)}^{a}+\overbrace{(\quad)}^{b}=\overbrace{(\quad)}^{b}+\overbrace{(\quad)}^{a}$

$\overbrace{(p)}^{a}+\overbrace{(q)}^{b}=\overbrace{(q)}^{b}+\overbrace{(p)}^{a}$

So we have $\boxed{p+q=q+p}$.

Notice that this has the same <u>structure</u> as the original identity $a+b=b+a$.

Using identities to rewrite expressions

Identities allow us to rewrite expressions by replacing an expression with a different equivalent expression (that may be simpler or more useful for graphing or solving something later).

Technically **we can <u>never</u> simply change or move something in an expression.**

Technically, **we can <u>only</u> rewrite expressions by <u>replacing some or all of the expression with an equivalent one</u>**.

Definition: Two expressions are *equivalent* if and only if they will <u>always</u> be equal, for <u>ALL possible values</u> of the variable(s). (If expressions contain multiple variables, this means for any possible combination of variable values.)

Examples:

For each of these expressions, we need to identify what is taking the place of the various variables in the identity. Just like in previous projects, what takes the place of the variables might not just be numbers, but could be more complex expressions that themselves contain variables or multiple terms.

We also have to be careful that <u>when choosing groupings within an expression, we always follow the order of operations</u>. So, as we add parentheses to group things together, we have to be sure that we only add parentheses when it doesn't change the structure imposed by the order of operations.

B) Use the identity $a+b=b+a$ to replace the original expression with an equivalent expression that has a different form:

$$2x+3y$$

$a=2x, \quad b=3y$

$a+b=b+a$

$\overbrace{(\quad)}^{a}+\overbrace{(\quad)}^{b}=\overbrace{(\quad)}^{b}+\overbrace{(\quad)}^{a}$

$\overbrace{(2x)}^{a}+\overbrace{(3y)}^{b}=\overbrace{(3y)}^{b}+\overbrace{(2x)}^{a}$

So: $\boxed{\mathbf{2x+3y=3y+2x}}$

C) Use the identity $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$ (whenever $c \neq 0$) to replace the original expression with an equivalent expression that has a different form:

$$\frac{2x^2+4y}{2}$$

$a=2x^2,\ \ b=4y,\ \ c=2$ (Since $2 \neq 0$, we know that $c \neq 0$)

$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$

$$\frac{\overbrace{(\ )}^{a}+\overbrace{(\ )}^{b}}{\overbrace{(\ )}^{c}} = \frac{\overbrace{(\ )}^{a}}{\overbrace{(\ )}^{c}}+\frac{\overbrace{(\ )}^{b}}{\overbrace{(\ )}^{c}}$$

$$\frac{\overbrace{(2x^2)}^{a}+\overbrace{(4y)}^{b}}{\overbrace{(2)}^{c}} = \frac{\overbrace{(2x^2)}^{a}}{\overbrace{(2)}^{c}}+\frac{\overbrace{(4y)}^{b}}{\overbrace{(2)}^{c}}$$

So: $\boxed{\mathbf{\frac{2x^2+4y}{2}=\frac{2x^2}{2}+\frac{4y}{2}}}$

D) Use the identity $x^n = \underbrace{x \cdot \cdots \cdot x}_{n-many\ times}$ (when n is a positive whole number) to replace the original expression with an equivalent expression that has a different form:

$$(3xy-4)^3$$

Important point! You may notice in this problem that we have an x in the identity and that we also have an x in the expression. This can be confusing. **These two x's are NOT the same, because they are coming from two totally <u>different</u> contexts. In the identity, the x is just standing in for ANY expression; whereas in the expression, the x may have a particular value or may represent a particular thing that is varying (for example: time, or distance).** In this case, it might be easier to <u>first</u> rewrite the identity with a different letter instead of using x. For example, we could rewrite the identity using the letter a in place of x to avoid the confusion:

$x^n = \underbrace{x \cdot \cdots \cdot x}_{n-many\ times} \quad \rightarrow \quad a^n = \underbrace{a \cdot \cdots \cdot a}_{n-many\ times}$

$a = 3xy-4,\quad n=3$ (Since 3 is a positive whole number, we know that n is a positive whole number)

$$\overbrace{(\ )}^{a}{}^{\overbrace{(\ )}^{n}} = \underbrace{\overbrace{(\ )}^{a} \cdot \cdots \cdot \overbrace{(\ )}^{a}}_{\overbrace{(\ )}^{n}-many\ times}$$

$$\overbrace{(3xy-4)}^{a}{}^{\overbrace{(3)}^{n}} = \underbrace{\overbrace{(3xy-4)}^{a} \cdot \cdots \cdot \overbrace{(3xy-4)}^{a}}_{\overbrace{(3)}^{n}-many\ times}$$

$$\overbrace{(3xy-4)}^{a}{}^{\overbrace{(3)}^{n}} = \underbrace{\overbrace{(3xy-4)}^{a} \cdot \overbrace{(3xy-4)}^{a} \cdot \overbrace{(3xy-4)}^{a}}_{\overbrace{(3)}^{n}-many\ times}$$

So: $\boxed{\mathbf{(3xy-4)^3=(3xy-4)\cdot(3xy-4)\cdot(3xy-4)}}$

Use the identity $x^0=1$ (whenever $x \neq 0$) to replace the original expression with an equivalent expression that has a different form: $\quad (2xy^2-3y)^0 \quad\quad (where\ 2xy^2-3y \neq 0)$

We could rewrite the identity $x^0=1$ (whenever $x \neq 0$) with a instead of x to avoid the confusion between the two x's (one in the identity and another in the expression):

$x^0=1$ (whenever $x \neq 0$) $\quad \rightarrow \quad a^0=1$ (whenever $a \neq 0$)

$a=2xy^2-3y$

$a^0=1$

$\overbrace{(\ )}^{a}{}^{0} = 1 \quad (as\ long\ as\ a \neq 0)$

$\overbrace{(2xy^2-3y)}^{a}{}^{0} = 1 \quad (because\ we\ know\ that\ 2xy^2-3y \neq 0)$

So: $\boxed{\mathbf{(2xy^2-3y)^0=1}}$

Now you try! For each of the following problems, use the given identity to rewrite the given expression in a different equivalent form. **Use the examples above as a model, writing out all of the same steps.**

1. Use the identity $a - b = a + -b$ to replace the original expression with an equivalent expression that has a different form:

$$2x - 3y$$

2. Use the identity $\frac{a \cdot b}{c} = a \cdot \frac{b}{c}$ (whenever $c \neq 0$) to replace the original expression with an equivalent expression that has a different form:

$$\frac{(2x^2)(4y)}{2}$$

3. Use the identity $nx = \underbrace{x + \cdots + x}_{n-many\ times}$ (when n is a positive whole number) to replace the original expression with an equivalent expression that has a different form:

$$3(xy - 4)$$

4. Use the identity $a(b + c) = ab + ac$ to replace the original expression with an equivalent expression that has a different form:

$$(2x - 1)(3x^2 + 7)$$

Does it matter whether an expression has the form of the right or the left side of an identity?

You may have noticed that in all the examples so far, we have taken something that has the structure of the left side of the identity and replaced it with whatever is on the right side of the identity. But we can also do the reverse, because:

If two things are equal, we can always replace one with the other! It doesn't matter which side of an equation is written on the left or the right—the equals sign in the middle just tells us that both sides are equal.

Examples:

F) Use the identity $x^{-n} = \frac{1}{x^n}$ (whenever $x \neq 0$) to replace the original expression with an equivalent expression that has a different form:

$$\frac{1}{\left(3p - \sqrt{r}\right)^8} \qquad (3p - \sqrt{r} \neq 0)$$

This expression has the structure of the **right** side of the identity $x^{-n} = \frac{1}{x^n}$ rather than the left side.
If it is helpful, we could rewrite the identity to look like this: $\frac{1}{x^n} = x^{-n}$ (whenever $x \neq 0$), because this is just another way of saying exactly the same thing—that the two sides are equal.
We can now use this identity to rewrite our expression $\frac{1}{(3p-\sqrt{r})^8}$:

$x = 3p - \sqrt{r},\ \ n = 8$
$\frac{1}{x^n} = x^{-n}$

$$\frac{1}{\overbrace{(\ )}^{x}{}^{\overbrace{(\ )}^{n}}} = \overbrace{(\ \ )}^{x}{}^{-\overbrace{(\ )}^{n}}$$

$$\frac{1}{\overbrace{(3p-\sqrt{r})}^{x}{}^{\overbrace{(8)}^{n}}} = \overbrace{(3p - \sqrt{r})}^{x}{}^{-\overbrace{(8)}^{n}}$$ (We know that $x \neq 0$ because $3p - \sqrt{r} \neq 0$.)

So $\boxed{\boldsymbol{\frac{1}{(3p-\sqrt{r})^8} = \left(3p - \sqrt{r}\right)^{-8}}}$

G) Use the identity $a(b + c) = ab + ac$ to replace the original expression with an equivalent expression that has a different form:

$$2x(3x^3y^5) + 2x(3y - 4x)$$

This expression has the structure of the **right** side of the identity $a(b + c) = ab + ac$ rather than the left side.
If it is helpful, we could rewrite the identity to look like this: $ab + ac = a(b + c)$, because this is just another way of saying exactly the same thing—that the two sides are equal.
We can now use this identity to rewrite our expression $2x(3x^3y^5) + 2x(3y - 4x)$:
$a = 2x,\ \ b = 3x^3y^5,\ \ \ c = 3y - 4x$
$ab + ac = a(b + c)$

$$\overbrace{(\ \ )}^{a}\overbrace{(\ \ )}^{b} + \overbrace{(\ \ )}^{a}\overbrace{(\ \ )}^{c} = \overbrace{(\ \ )}^{a}\left(\overbrace{(\ \ )}^{b} + \overbrace{(\ \ )}^{c}\right)$$

$$\overbrace{(2x)}^{a}\overbrace{(3x^3y^5)}^{b} + \overbrace{(2x)}^{a}\overbrace{(3y - 4x)}^{c} = \overbrace{(2x)}^{a}\left(\overbrace{(3x^3y^5)}^{b} + \overbrace{(3y - 4x)}^{c}\right)$$

So $\boxed{\boldsymbol{(2x)(3x^3y^5) + (2x)(3y - 4x) = (2x)\left((3x^3y^5) + (3y - 4x)\right)}}$

Now you try! For each of the following problems, use the given identity to rewrite the given expression in a different equivalent form. **Use the examples above as a model, writing out all of the same steps.**

5. Use the identity $a(b+c) = ab + ac$ to replace the original expression with an equivalent expression that has a different form:

$$3xy(a^2b^3) + 3xy(a-1)$$

6. Use the identity $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ (whenever $c \neq 0$) to replace the original expression with an equivalent expression that has a different form:

$$\frac{9x + 3\sqrt{x^3}}{3}$$

7. Use the identity $a \cdot \frac{1}{b} = \frac{a}{b}$ (whenever $b \neq 0$) to replace the original expression with an equivalent expression that has a different form:

$$\frac{pqr - 1}{2pq^2r}$$

8. Use the identity $ac + bc = (a+b)c$ to replace the original expression with an equivalent expression that has a different form:

$$(x+7)(2xy-1)$$