Project 2: How Parentheses and the Order of Operations Impose Structure on Expressions

Parentheses show us how things should be grouped together. The sole purpose of parentheses in algebraic expressions is to impose a particular structure, to show us which things are grouped together. Things together within a set of parentheses are treated as a single unit. So, for example:

- $(2x+1)-(3x+7)$ Here we treat $(2x+1)$ and $(3x+7)$ each as single units, the whole expression $(3x+7)$ is being subtracted from the whole first expression $(2x+1)$.
- $3(xy)^3$ Here we treat (xy) as a single unit. The whole expression (xy) is being raised to the third power.
- $3(2x^2-3x+1)$ Here we treat $(2x^2-3x+1)$ as a single unit. The whole expression $(2x^2-3x+1)$ is being multiplied by 3.

Implied or hidden parentheses (and other hidden symbols)

There are a number of conventions for writing algebraic expressions and equations that allow us to leave out parentheses or other symbols. Conventions are just rules that mathematicians have made up to allow us to write certain things quickly or simply. They are like abbreviations—using conventions, we can write something using fewer parentheses or symbols, and this makes the expressions faster to write and easier to read. However, in order for this to work, we have to memorize the conventions so that we know how to read mathematical expressions correctly. Here are some examples of some common math conventions that we need to know:

I. Two things written next to each other (except for two numbers) means multiplication. So, for example:

- $-2x=-2\cdot x$
- $pq=p\cdot q$
- $2ab^3c=2\cdot a\cdot b^3\cdot c$
- $(3x^2-4)(x^2+5x-9)=(3x^2-4)\cdot(x^2+5x-9)$
- $3\sqrt{4}=3\cdot\sqrt{4}$
- $(2)(3)=(2)\cdot(3)$
  Note: 23 is NOT two times three, but rather the number twenty-three.
  Note: The parentheses here do NOT mean multiplication—the fact that the **two factors are written next to each other is what indicates multiplication**. The parentheses are included here to make clear that this is 2 times 3 and NOT the number 23. It is very important not to confuse parentheses with multiplication. For example, even though there are parentheses, there is no multiplication in the expression $(x+1)-(x+7)$.

Now you try! Put in the missing multiplication sign (use a dot instead of an x):
For each of the following questions, rewrite the expression using the multiplication dot in between each pair of factors that are being multiplied together (you can use the examples above as a guide):

1. xy
2. $4p$
3. $2\sqrt{3}$
4. $(-2)(7)$
5. $(2x-1)(3x^2+7)$
6. $-3pqr^2$

II. Subtraction: the minus sign applies to whatever term directly follows it. If a parenthesis is just to the right of the minus sign, then everything inside the parentheses must be subtracted. If a term that begins with a number or variable is to the right of the minus sign, then only that term is subtracted. So, for example:

- $(2x+1)-(3x+7)$ The whole $(3x+7)$ is subtracted.
- $3x^2-2x+4x+5=3x^2-(2x)+4x+5$ Just the $2x$ is subtracted. If we want to put a parenthesis in front of the $2x$, we MUST close the parenthesis IMMEDIATELY AFTER the $2x$. We CANNOT, for example, put a set of parentheses around all of the terms $2x+4x+5$, because then we would be subtracting ALL of these terms—but in this problem, we are ONLY supposed to be subtracting the $2x$.

Now you try! Circle what is being subtracted:
For each of the following questions, CIRCLE whatever is actually being subtracted (use examples above as a guide):

7. $3x^2+7-2x+1$

8. $(3x^2+7)-(2x+1)$

9. $2-xyz$

10. $(2-x)yz$

III. Exponents only apply to whatever is directly to the left of the exponent.
If a parenthesis is just to the left of the exponent, then everything inside the parentheses is the base.
If a number or variable is to the left of the exponent, then only that number or variable is the base—if there is a negative sign out front, it is NOT part of the base, unless the whole thing is in parentheses. So, for example:

- $-2^2=-(2)^2$ Only 2 is squared. (So we would get: $-(2\cdot 2)=-4$.)
- $(-2)^2$ Negative 2 is squared. (So we would get: $(-2)(-2)=-4$.)
- $3xy^3=3x(y)^3$ Only y is raised to the third power. (So this is really $3xyyy$.)
- $3(xy)^3$ The whole expression xy is raised to the third power, but 3 is not. (So this is really $3xyxyxy$.)
- $(3xy)^3$ The whole expression $3xy$ is raised to the third power. (So this is really $(3xy)(3xy)(3xy)$.)
- $x-y^2=x-(y)^2$ Only y is squared, not the x, and not the negative sign. (So this is really $x-(y)(y)$.)
- $(x-y)^2$ The whole expression $x-y$ is squared. (So this is really $(x-y)(x-y)$.)

Now you try! Circle the base of the exponent:
For each of these questions CIRCLE whatever is being raised to the exponent (use examples above as a guide):

11. $-4x+7^3$

12. $(-4x+7)^3$

13. $-4x^3+7$

14. $(-4x)^3+7$

15. $-(4x)^3+7$

16. -4^3x+7

17. $(-4)^3x+7$

18. $-4(x+7)^3$

IV._Fraction bars, bars, radical signs, and each side of an equation function like parentheses. For fractions, the top and bottom of the fraction are really inside their own parentheses, even though we don't write them out. Anything under a radical sign is really inside parentheses, even though we don't write that out either. Anything on one or the other side of an equation is also inside parentheses, even though we don't write them. So, for example:

- $\frac{2-x}{x} = \frac{(2-x)}{x}$ or $(2-x)/x$ or $(2-x) \div x$
- $\frac{3a^2+b}{a-b} = \frac{(3a^2+b)}{(a-b)}$ or $(3a^2+b)/(a-b)$ or $(3a^2+b) \div (a-b)$
- $\sqrt{x^2+y^2} = \sqrt{(x^2+y^2)}$
- $2x-7 = 1-2x \quad \leftrightarrow \quad (2x-7) = (1-2x)$
  (We use the double-sided arrow symbol to show that two equations are equivalent, since another equals sign would be confusing.)
- $x(x-5) = x+1 \quad \leftrightarrow \quad \big(x(x-5)\big) = (x+1)$

<u>Now you try! Put in the missing parentheses:</u>
For each of the following questions, rewrite the expression by putting parentheses around whatever is the numerator or denominator of a fraction, or whatever is under a radical (use the examples above as a guide):

19. $\frac{-4x+7}{2x}$

20. $\frac{xy^2-xyz}{xy^2-x^2y}$

21. $\sqrt{3xy^2z^3}$

22. $\sqrt[3]{2x^2-y^2}$

23. $3x^2+7x-1 = x-1$

24. $y = 2x-7$

V. How do we group things in an expression? The order of operations imposes structure on expressions

You may have already learned the order of operations as a set of rules that tells us what order to do certain operations in when making calculations with numbers. However, when thinking algebraically, we need to think about the order of operations as a way of **imposing structure on expressions**, instead of as an order of calculating:

1. The order of operations is just a set of arbitrary conventions about where we should assume the hidden parentheses are (just like the conventions listed above); and
2. The order of operations imposes a structure on algebraic expressions, telling us **which parts of the expression we should clump together and treat as a single unit**.

While you may have learned more complex rules for the order of operations in the past, **there are really only two basic conventions that we need to keep in mind in order to remember the order of operations:**

A. **Things inside parentheses are always treated as a single unit, from the inside out.** This includes implied parentheses that we have to assume are there but that aren't written out, like those around the top and bottom of a fraction, or those around what is under a radical sign.

B. **If there are no parentheses to tell us otherwise, a string of multiplication (including exponents) and division combined forms a single term, which we view as a single unit. If multiplication/exponents/division are mixed with addition/subtraction, then we must treat each term (in between the addition/subtraction) as a single unit.**

You may recall the word *term* being used in previous algebra classes. This is just the word for a bunch of stuff that is clumped together by a mixture of exponents/multiplication/division. Terms are treated as single units because of the order of operations (in which multiplication is done before addition/subtraction) and are separated from each other by addition or subtraction.

How does this play out in actual expressions? Consider the following examples:

- $2x^2 - (xy)^4$ Here there are several structures.
    1. **Circle any real or invisible parentheses in BLACK.** This expression has only one set of parentheses, in the term $(xy)^4$, so we circle those: $2x^2 - \boxed{(xy)}^4$
    2. **Circle the base of each exponent in RED.** In the term $2x^2$, only the *x* is next to the exponent, so that is the base: $2(x)^2$. In the term $(xy)^4$, the parentheses are next to the exponent, so the whole expression (xy) is raised to the fourth power, so we replace the black circle with a red one. So we have: $2\boxed{x}^2 - \boxed{(xy)}^4$
    3. **Circle every group of things that are just being multiplied and/or divided in BLUE.** There are two groups of things being multiplied in this expression (separated by a subtraction sign). $2x^2$ is a group of things being multiplied together, which we could write as $2 \cdot x^2$. Similarly, $(xy)^4$ is a group of things being multiplied together (remember that exponents also indicate multiplication—in this case, that we should multiply xy by itself four times), so we could write this as $(x \cdot y)^4$. So we have: $\boxed{2\boxed{x}^2} - \boxed{\boxed{(xy)}^4}$
- $\frac{3(xy^2)^3}{2x-xy}$ Here there are also several structures:
    1. **Circle any real or invisible parentheses in BLACK.** This expression has only one set of parenthesis, but it also has hidden parentheses around the top and bottom of the fraction, so we circle each of these: $\frac{\boxed{3\boxed{(xy^2)}^3}}{\boxed{2x-xy}}$
    2. **Circle the base of each exponent in RED.** In the top of the fraction, we have two exponents. The next to the square exponent inside the parentheses is only a *y*, so *y* is the base of this exponent: $x\boxed{y}^2$. There is also an exponent outside the parentheses—right next to the parentheses in fact, which means that everything inside the parentheses is the base, so we replace the black circle with a red one: $\boxed{(x\boxed{y}^2)}^3$. So we have: $\frac{\boxed{3\boxed{(x\boxed{y}^2)}^3}}{\boxed{2x-xy}}$
    3. **Circle every group of things that are just being multiplied and/or divided in BLUE.** There are several groups of things being multiplied together in this expression. Let's look at the top and bottom of the fraction separately: At the top, everything is being multiplied together, so we can replace the black circle at the top of the fraction with a blue one: $\boxed{3\boxed{(x\boxed{y}^2)}^3}$. On the bottom, we have two terms, each of which contains only multiplication, separated by a subtraction sign: $\boxed{2x} - \boxed{xy}$. Putting this all together gives us: $\frac{\boxed{3\boxed{(x\boxed{y}^2)}^3}}{\boxed{\boxed{2x}-\boxed{xy}}}$

<u>Now you try! Marking up structure:</u>
For each of the following questions:

a) **Circle any visible or hidden parentheses in BLACK.**
b) **Circle the base of each exponent in RED.**
c) **Circle every group of things that are being multiplied or divided in BLUE.**

25. $8x^2 - xy$

26. $2x^5y^{-4}$

27. $\frac{2x^2-4x}{2x}$

28. $(3x)^2 - 2x + 5x^2$

29. $(7x^2 - 6x) - (4x^2 - 3x)$

30. $(4x - 2)(x^2 - 5x + 3)$

31. $\frac{20x^9-30x^4+10x^3}{-10x^3}$

The Importance of Parentheses: When does adding or removing them change the structure?

In past classes, you may have thought about adding or removing parentheses from expressions or equations. In this class, we will NOT think this way—instead, we will think about **replacing one expression or equation with an equivalent one**. To see why this is important, let's look at how different expressions with and without parentheses have different meanings, and how "removing" or "adding" in parentheses can change the structure.

<u>**Example 1.** We consider the two expressions $2x \cdot (x - 7)$ and $2x \cdot x - 7$.</u>
First we circle the two things being **multiplied** in each expression: $\boxed{2x} \cdot \boxed{(x-7)}$ and $\boxed{2x} \cdot \boxed{x} - 7$

In the expression $2x \cdot (x - 7)$, $2x$ is being multiplied by $x - 7$. Notice that because of the parentheses, we think of $x - 7$ as a single unit, so it is better to write that $2x$ is being multiplied by $(x - 7)$. Now consider the **different** but similar-looking expression $2x \cdot x - 7$. In this expression, the $2x$ is only being multiplied by the x.

So these two expressions have **different structures**, because **different** sub-expressions are being multiplied together in each expression. So can we replace one of these expressions with the other? Probably not. Sometimes we may get lucky, and similar-looking expressions with different structures may turn out to be equivalent—in other words, it may turn out that they always have the same value no matter what we plug in for *x* (or other variables in the expression). But unless we can first establish that they will be equivalent, we can **NOT replace** one equation with a different equation that has a **different structure**.

<u>**Example 2.** We consider the two expression $4x^3 - 3x^2 + 2x - 1$ and $4x^3 - (3x^2 + 2x) - 1$.</u>
First we circle the things being **subtracted** in each expression: $4x^3 - \boxed{3x^2} + 2x - \boxed{1}$ and $4x^3 - \boxed{(3x^2 + 2x)} - \boxed{1}$

In the expression $4x^3 - 3x^2 + 2x - 1$, first $3x^2$ and then later 1 is being subtracted. However, in $4x^3 - (3x^2 + 2x) - 1$, first $(3x^2 + 2x)$ and then later 1 is being subtracted. What is being subtracted in each expression is **different**, so these two expressions have a **different structure**. So we cannot replace one of these expressions with the other unless we are able to construct a convincing argument about why they are equivalent (which we don't know how to do yet). So we cannot replace one expression with the other.

Example 3. We consider the two expression $4x^3 + 3x^2 + 2x - 1$ and $4x^3 + (3x^2 + 2x) - 1$.

First we circle the things being **subtracted** in each expression: $4x^3 + \boxed{3x^2} + \boxed{2x} - 1$ and $4x^3 + \boxed{(3x^2 + 2x)} - 1$

In the expression $4x^3 - 3x^2 + 2x - 1$, first $3x^2$ and then later $2x$ is being added to the $4x^3$. However, in $4x^3 + (3x^2 + 2x) - 1$, $(3x^2 + 2x)$ is being added to the $4x^3$ all at once. What is being added in each expression is slightly **different**, so these two expressions have a **different structure**. So we cannot replace one of these expressions with the other unless we are able to construct a convincing argument about why they are equivalent (which we don't know how to do yet). You may notice that adding first $3x^2$ and then $2x$ is an equivalent process as adding $(3x^2 + 2x)$ all at once, because adding things is just like throwing them all together into a single container and then counting up the total—so adding things one at a time or all at once **will not change the result, regardless of what *x* stands for**. So in this case, it turns out the that two expressions would in fact be equivalent. But this isn't obvious right away, and before we can conclude that they are equivalent, we need to make a clear argument for why adding $3x^2$ and then $2x$ separately will always produce the same result as adding $(3x^2 + 2x)$ all at once—we will look at how to make this kind of argument in the next project. For now, **we just notice that the structure is different, and we note exactly how it is different**, so that we can think about whether these different structures will be equivalent.

Now you try! For each of the following groups of expressions, answer the questions to determine whether and how the structures are different.

Expressions	**What is being multiplied in the first expression?**	**What is being multiplied in the second expression?**	**Do these two expressions have the same structure?**	**Are these two expressions equivalent?**
32. $2 \cdot (3x - 5)$ vs $2 \cdot 3x - 5$			☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**	☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**
33. $(2x + 1) \cdot (3x - 7)$ vs $2x + 1 \cdot 3x - 7$			☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**	☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**
34. $(x + 2) - 3(x + 7)$ vs $x + 2 - 3x + 7$			☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**	☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**
35. $\frac{2x \cdot 4y}{2}$ vs $2x \cdot \frac{4y}{2}$			☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**	☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**

Expressions	What is being <u>subtracted</u> in the first expression?	What is being <u>subtracted</u> in the second expression?	Do these two expressions have the <u>same structure</u>?	Are these two expressions <u>equivalent</u>?
36. $2x-(3x+6)$ vs $2x-3x+6$			☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**	☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**
37. $(2x+1)-(3x-7)$ vs $2x+1-3x-7$			☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**	☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**
38. $(x+2)-3(x+7)$ vs $x+2-3x+7$			☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**	☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**
39. $2x^3+3x^2-4x+6$ vs $2x^3+3x^2-(4x+6)$			☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**	☐ **don't know** ☐ **no** ☐ **yes** **Why or why not?**

Expressions	What is being <u>added</u> in the first expression?	What is being <u>added</u> in the 2nd expression?	Do these have the <u>same structure</u>?	Are these two expressions <u>equivalent</u>?
40. $2x + (3x - 6)$ vs $2x + 3x - 6$			☐ don't know ☐ no ☐ yes Why or why not?	☐ don't know ☐ no ☐ yes Why or why not?
41. $(2x + 1) + (3x - 7)$ vs $2x + 1 + 3x - 7$			☐ don't know ☐ no ☐ yes Why or why not?	☐ don't know ☐ no ☐ yes Why or why not?
42. $2x^3 + 3x^2 + 4x + 6$ vs $2x^3 + 3x^2 + (4x + 6)$			☐ don't know ☐ no ☐ yes Why or why not?	☐ don't know ☐ no ☐ yes Why or why not?
43. $\frac{2+4y}{2}$ vs $2 + \frac{4y}{2}$			☐ don't know ☐ no ☐ yes Why or why not?	☐ don't know ☐ no ☐ yes Why or why not?

Expressions	What is the <u>base of the exponent</u> in the 1st expression?	What is the <u>base of the exponent</u> in the 2nd expression?	Do these have the <u>same structure</u>?	Are these two expressions <u>equivalent</u>?
44. $(3x - 5)^2$ vs $3x - 5^2$			☐ don't know ☐ no ☐ yes Why or why not?	☐ don't know ☐ no ☐ yes Why or why not?
45. $5x \cdot (2x^2y)^{-3}$ vs $5x \cdot 2x^2y^{-3}$			☐ don't know ☐ no ☐ yes Why or why not?	☐ don't know ☐ no ☐ yes Why or why not?
46. -2^3 vs $(-2)^3$			☐ don't know ☐ no ☐ yes Why or why not?	☐ don't know ☐ no ☐ yes Why or why not?