

Project 8: Rewriting Equations

We can use the following properties to rewrite equations (to put them in a particular form like $y - mx + b$, or to solve them, or to combine multiple equations into a single equation).

Properties that apply to equations specifically:	
$a = b \leftrightarrow b = a$	
$a = b \leftrightarrow a + c = b + c \quad a < b \leftrightarrow a + c < b + c$	
$a = b \leftrightarrow a \cdot c = b \cdot c \quad (\text{whenever } c \neq 0) \quad a < b \leftrightarrow \begin{cases} a \cdot c < b \cdot c & \text{if } c > 0 \\ a \cdot c > b \cdot c & \text{if } c < 0 \end{cases}$	
$a = b \leftrightarrow \frac{a}{c} = \frac{b}{c} \quad (\text{whenever } c \neq 0) \quad a < b \leftrightarrow \begin{cases} \frac{a}{c} < \frac{b}{c} & \text{if } c > 0 \\ \frac{a}{c} > \frac{b}{c} & \text{if } c < 0 \end{cases}$	
$a = b \text{ and } c = d \leftrightarrow a + c = b + d \quad a = b \text{ and } c = d \leftrightarrow na + mc = nb + md$ for any numbers n and m	
$ab = 0 \leftrightarrow a = 0 \text{ or } b = 0$	
$ax^2 + bx + c = 0 \leftrightarrow ax^2 + b_1x + b_2x + c = 0$ where <ul style="list-style-type: none"> * b_1, b_2 are whole numbers * $b_1 \cdot b_2 = ac$ * $b_1 + b_2 = b$ 	
Properties for expressions, that can also be used on expressions inside equations:	
$a(b + c) = ab + ac \quad (a + b)c = ac + bc$	
$a(x_1 + \dots + x_n) = ax_1 + \dots + ax_n \quad (x_1 + \dots + x_n)c = x_1c + \dots + x_nc$	
$x^n = \underbrace{x \cdot \dots \cdot x}_{n\text{-many times}}$	
$x^{-n} = \frac{1}{x^n} \quad (\text{whenever } x \neq 0)$	
$\frac{x}{1} = x$	
$\frac{x}{x} = 1 \quad (\text{whenever } x \neq 0)$	
$\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{d} \quad (\text{whenever } c, d \neq 0) \quad \frac{x_1 \cdot \dots \cdot x_n}{y_1 \cdot \dots \cdot y_n} = \frac{x_1}{y_1} \cdot \dots \cdot \frac{x_n}{y_n} \quad (\text{whenever } y_1, \dots, y_n \neq 0)$	
$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \quad (\text{whenever } c \neq 0) \quad \frac{x_1 + \dots + x_n}{y} = \frac{x_1}{y} + \dots + \frac{x_n}{y} \quad (\text{whenever } y \neq 0)$	
$\sqrt{x^2} = x \quad (\text{whenever } x \geq 0)$	
$\sqrt{ab} = \sqrt{a}\sqrt{b}$	
$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad (\text{whenever } b \neq 0)$	

What does it mean to solve an equation for a specific variable?

Often we think of solving an equation as a process that leads to a specific answer, and sometimes that happens when we solve an equation. But that is not really what it means. **Solving an equation just means that we want to isolate the given variable until it is by itself on only one side of the equation.**

So, for example, this equation is NOT solved for x : $2x + 6 = x$ because we have x 's on both sides of the equation, and also because the x on the left side of the equation is not by itself.

On the other hand, this equation IS solved for p : $p = \frac{nr}{v}$ because the p is by itself on one side of the equation. The fact that the other side of the equation still contains a bunch of other variables instead of a number is irrelevant.

Which of the following equations have been solved for x , and which have not? Circle each one that has been solved for x , and cross out any that have not yet been solved for x .

1. $x = 2$
2. $-3 = x$
3. $x = 2y + 7$
4. $x = 3 + x^2 - 4$
5. $x = \frac{pqr}{s}$
6. $x < 9$
7. $8 \geq x$
8. $x = \frac{8}{5}$
9. $x = \sqrt{5}$

How do we solve an equation for a specific variable?

Just like with simplifying or rewriting expression, the process of solving an equation for a specific variable is really just a process of replacing an equation with a different equation that is equivalent to it.

How can we tell if two equations are equivalent? **Two equations (or inequalities, or systems of equations) are equivalent if they have the same solution set.**

The solution set is the set of all numbers that could be put in for the variables that would make the equation true.

So, for example:

- The two equations $x - 5 = 2$ and $x^2 + 49 = 14x$ are equivalent because they both have the same solution set $x = 7$. As it turns out, 7 is the only value that we can substitute in for x that will make the first or the second equation true. Even though these two equations don't look at all similar, it turns out they are actually equivalent because they have the same solution set.
- The two equations $x + y = z$ and $(x + y) + 5 = z + 5$ are equivalent because they both have the same solution set: any combination of values that we substitute in for x, y, z in the first equation will also work in the second equation, since the second equation is the same as the first with only a 5 added to each side of the equation.
- **To show that two equations are equivalent, we can NOT use the equals sign.** Since equations already contain an equals sign, adding a third equals sign between two equations would be confusing. Instead, we use the double-arrow, like this:

$$x - 5 = 2 \leftrightarrow x^2 + 49 = 14x$$

$$x + y = z \leftrightarrow (x + y) + 5 = z + 5$$

So, to rewrite an equation, we simply need to replace it with an equivalent equation. The most systematic way to do this is to use properties (or identities) to substitute some or all of the equation for an equivalent equation.

This is just like what we did with expressions, except with equations we do have to think carefully about where we put the equals sign.

Example: Put the equation $-3(x - 2) = y + 1$ into the form $y = mx + b$ (where m, b are any numbers), and identify the values of m and b .

First, just like with simplifying expressions, I will rewrite all subtraction as adding a negative:

$$-3(x - 2) = y + 1 \leftrightarrow -3(x + -2) = y + 1$$

Then, just like with simplifying expressions, I will try to rewrite the equation without parentheses, in this case using the distributive property:

Using $a = -3$, $b = x$, $c = -2$ for $a(b + c) = ab + ac$ yields:

$$-3(x + -2) = y + 1 \leftrightarrow -3x + (-3)(-2) = y + 1$$

Multiplying $-3 \cdot -2$ gives us:

$$-3x + (-3)(-2) = y + 1 \leftrightarrow -3x + 6 = y + 1$$

Now, we need this to be in the form $y = mx + b$ (where m, b are any numbers). So we start by getting the side with the y in it on the left side and the side with the x in it on the right side by using the property $a = b \leftrightarrow b = a$:

Using $a = -3x + 6$, $b = y + 1$ for $a = b \leftrightarrow b = a$ yields:

$$-3x + 6 = y + 1 \leftrightarrow y + 1 = -3x + 6$$

Now, I almost have what I need, but I need for the $+1$ that is on the left side to go elsewhere (because in the form $y = mx + b$, the y is by itself on the left side). I look at the possibly properties that I could use, and I notice that if I were to add -1 to the $+1$ these would leave me with a zero on the left side (which when added to the y would just give me y on that side). I notice that I have the following property which tells me that I can add anything I want to one side of an equation as long as I also add it to the other side: $a = b \leftrightarrow a + c = b + c$

So, using $a = y + 1$, $b = -3x + 6$ and choosing $c = -1$ for $a = b \leftrightarrow a + c = b + c$ yields:

$$y + 1 = -3x + 6 \leftrightarrow (y + 1) + -1 = (-3x + 6) + -1$$

If I think of each term as a single unit, I can see that all terms on the left are being added, and all terms on the right are being added, so I can rewrite the expressions on both the left and the right side of the equation without the parentheses using the identity $(a + b) + c = a + b + c$:

$$(y + 1) + -1 = (-3x + 6) + -1 \leftrightarrow y + 1 + -1 = -3x + 6 + -1$$

Now once again since all the terms on the left and all the terms on the right are being added, I can combine them in any order using the identity $a + b = b + a$. So I can add $1 + -1$ first on the left and $6 + -1$ first on the right:

$$y + 1 + -1 = -3x + 6 + -1 \leftrightarrow y + 0 = -3x + 5$$

I almost have what I need. I just need to rewrite the equation without the 0 on the left side. But I have an identity that says that $x + 0 = x$. So if I let $y = x$, I can rewrite this equation one last time by replacing the expression $y + 0$ on the left side of the equation with the expression y instead:

$$y + 0 = -3x + 5 \leftrightarrow y = -3x + 5$$

Now the equation has the form $y = mx + b$, and we have $m = -3$, $b = 5$.

Notice two important things:

- 1) Every time I **replaced the equation with an equivalent expression using an identity**.
- 2) When using the identities I was careful to **use parentheses every time for the variables** (I could then only rewrite the equation without the parentheses afterwards if I could find an identity that showed me how to write an equivalent equation without the parentheses).

Now you try! For each of the following equations, put them into the requested form:

10. $2x - 3y = 6$	Put into $y = mx + b$ form (where m, b are any numbers) and give the values of m, b :
11. $y - 2 = -1(x - 3)$	Put into $y = mx + b$ form (where m, b are any numbers) and give the values of m, b :
12. $y = 5$	Put into $y = mx + b$ form (where m, b are any numbers) and give the values of m, b :
13. $2x - x^2 = 3 - x + 2x^2$	Put into $ax^2 + bx + c = 0$ form (where a, b, c are any numbers) and give the values of a, b, c :

14. $x^2 = 5 - x$	Put into $ax^2 + bx + c = 0$ form (where a, b, c are any numbers) and give the values of a, b, c :
15. $x^2 = 5$	Put into $ax^2 + bx + c = 0$ form (where a, b, c are any numbers) and give the values of a, b, c :

Use the properties on the first page of the project to solve each of the following equations, by:

- 1) first simplifying each side of the equation or inequality,
- 2) and then rewriting as needed to get the indicated variable by itself on one side.
- 3) Then check your work by substituting the solution into the original equation.

Original equation	Solve for the variable indicated	Check your work
<p>Example:</p> <p>Solve for x:</p> $\frac{2x}{2} + \frac{4}{3} = \frac{5}{6}$	<p>The fractions are making this equation more complicated than we would like, so my first goal is to rewrite the equation without them. Fractions are just division, so one way to rewrite the equation is to multiply both sides by the least common multiple of all the denominators—6 is a multiple of 2, and 3, and 6. So if I multiply both sides of the equation by 6 using the identity $a = b \leftrightarrow a \cdot c = b \cdot c$, this should allow me to rewrite the equation without the fractions:</p> <p>Here $a = \frac{2x}{2} + \frac{4}{3}$, $b = \frac{5}{6}$, $c = 6$:</p> $\frac{2x}{2} + \frac{4}{3} = \frac{5}{6} \leftrightarrow 6 \cdot \left(\frac{2x}{2} + \frac{4}{3}\right) = 6 \cdot \left(\frac{5}{6}\right)$ $\leftrightarrow 6 \cdot \left(\frac{2x}{2}\right) + 6 \cdot \left(\frac{4}{3}\right) = 6 \cdot \left(\frac{5}{6}\right)$ $\leftrightarrow \frac{6}{1} \cdot \left(\frac{2x}{2}\right) + \frac{6}{1} \cdot \left(\frac{4}{3}\right) = \frac{6}{1} \cdot \left(\frac{5}{6}\right)$ $\leftrightarrow \frac{6 \cdot 2x}{1 \cdot 2} + \frac{6 \cdot 4}{1 \cdot 3} = \frac{6 \cdot 5}{1 \cdot 6}$ $\leftrightarrow \frac{12x}{2} + \frac{24}{3} = \frac{30}{6}$ $\leftrightarrow \frac{12}{2}x + \frac{24}{3} = \frac{30}{6}$ $\leftrightarrow 6x + 8 = 5$ <p>Now that we no longer have fractions to deal with, we proceed to try to get x by itself on one side by thinking about what parts are “in the way”. The 8 needs to be rewritten: it is currently being added in the equation, so we add a negative 8 to both sides of the equation so that we get 0 on the left instead of 8, using the identity $a = b \leftrightarrow a + c = b + c$, and letting $a = 6x + 8, b = -3, c = -8$:</p> $\leftrightarrow (6x + 8) + -8 = (5) + -8$ $\leftrightarrow 6x + (8 + -8) = -3$ $\leftrightarrow 6x + 0 = -3$ $\leftrightarrow 6x = -3$ <p>Now we need to rewrite the 6 somehow—it is currently being multiplied in the equation, so we need to divide both sides of the equation by 6 so that we get 1 on the left instead of 6, using the identity $a = b \leftrightarrow \frac{a}{c} = \frac{b}{c}$ (whenever $c \neq 0$) and letting $a = 6x, b = -3, c = 6$:</p> $\leftrightarrow 6x = -3$ $\leftrightarrow \frac{6x}{6} = \frac{-3}{6}$ $\leftrightarrow \frac{6}{6}x = \frac{-3}{6}$ $\leftrightarrow 1x = \frac{-1}{2}$ $\leftrightarrow x = \frac{-1}{2}$	$x = -\frac{1}{2} \text{ and}$ $\frac{2x}{2} + \frac{4}{3} = \frac{5}{6}$ $\leftrightarrow \frac{2\left(-\frac{1}{2}\right)}{2} + \frac{4}{3} = \frac{5}{6}$ $\leftrightarrow \frac{\frac{2\left(-\frac{1}{2}\right)}{1}}{2} + \frac{4}{3} = \frac{5}{6}$ $\leftrightarrow \frac{\frac{2 \cdot 1}{1 \cdot 2}}{2} + \frac{4}{3} = \frac{5}{6}$ $\leftrightarrow \frac{-\frac{2}{2}}{2} + \frac{4}{3} = \frac{5}{6}$ $\leftrightarrow \frac{-1}{2} + \frac{4}{3} = \frac{5}{6}$ $\leftrightarrow \frac{-1}{2} \cdot \frac{3}{3} + \frac{4}{3} \cdot \frac{2}{2} = \frac{5}{6}$ $\leftrightarrow \frac{-1 \cdot 3}{2 \cdot 3} + \frac{4 \cdot 2}{3 \cdot 2} = \frac{5}{6}$ $\leftrightarrow \frac{-3}{6} + \frac{8}{6} = \frac{5}{6}$ $\leftrightarrow \frac{-3+8}{6} = \frac{5}{6} \text{ TRUE! } \checkmark$

16. $2(7 - a) = 6a - 2$

Solve for a :

17. $a = 6b + 2c$

Solve for b :

18. $3x - 4y = 8$

Solve for y :

19. $\frac{n+3}{2} = \frac{n+3}{3}$

Solve for n :

20. $6p - 1 = 5 - 2(3p + 3)$	Solve for p :	