Project 8: Rewriting Equations

We can use the following properties to rewrite equations (to replace them with equivalent equations that have a particular form like $y - mx + b$, or to solve them—which is really just another process of "rewriting" by replacing one equation with a sequence of equivalent ones, or to combine information from multiple equations into a single equation).

Properties that apply to equations specifically:
$a = b \leftrightarrow b = a$
$a = b \leftrightarrow a + c = b + c \qquad a < b \leftrightarrow a + c < b + c$
$a = b \leftrightarrow a \cdot c = b \cdot c \quad (whenever\ c \neq 0) \qquad a < b \leftrightarrow \begin{cases} a \cdot c < b \cdot c & if\ c > 0 \\ a \cdot c > b \cdot c & if\ c < 0 \end{cases}$
$a = b \leftrightarrow \frac{a}{c} = \frac{b}{c} \quad (whenever\ c \neq 0) \qquad a < b \leftrightarrow \begin{cases} \frac{a}{c} < \frac{b}{c} & if\ c > 0 \\ \frac{a}{c} > \frac{b}{c} & if\ c < 0 \end{cases}$
$a = b\ and\ c = d \leftrightarrow a + c = b + d \qquad a = b\ and\ c = d \leftrightarrow na + mc = nb + md$ $for\ any\ numbers\ n\ and\ m$
$ab = 0 \leftrightarrow a = 0\ or\ b = 0$
$ax^2 + bx + c = 0 \leftrightarrow ax^2 + b_1x + b_2x + c = 0 \quad where$ $* b_1, b_2\ are\ whole\ numbers$, $* b_1 \cdot b_2 = ac$, $* b_1 + b_2 = b$
Properties for expressions, that can also be used on expressions inside equations:
$a(b + c) = ab + ac \qquad (a + b)c = ac + bc$ $a(x_1 + \cdots + x_n) = ax_1 + \cdots + ax_n \qquad (x_1 + \cdots + x_n)c = x_1c + \cdots + x_nc$
$x^n = \underbrace{x \cdot \ldots \cdot x}_{n-many\ times}$
$x^{-n} = \frac{1}{x^n} \quad (whenever\ x \neq 0)$
$\frac{x}{1} = x$
$\frac{x}{x} = 1 \quad (whenever\ x \neq 0)$
$\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{d} \quad (whenever\ c, d \neq 0) \qquad \frac{x_1 \cdot \ldots \cdot x_n}{y_1 \cdot \ldots \cdot y_n} = \frac{x_1}{y_1} \cdot \ldots \cdot \frac{x_n}{y_n} \quad (whenever\ y_1, \ldots, y_n \neq 0)$
$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \quad (whenever\ c \neq 0) \qquad \frac{x_1 + \cdots + x_n}{y} = \frac{x_1}{y} + \cdots + \frac{x_n}{y} \quad (whenever\ y \neq 0)$
$\sqrt{x^2} = x \quad (whenever\ x \geq 0)$
$\sqrt{ab} = \sqrt{a}\sqrt{b}$
$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad (whenver\ b \neq 0)$

<u>What does it mean to solve an equation for a specific variable?</u>
Often we think of solving an equation as a process that leads to a specific answer, and sometimes that happens when we solve an equation. But that is not really what it means. **Solving an equation just means that we want to replace an equation with a sequence of equivalent equations, until we have an equation in which the given variable is <u>by itself</u> on only <u>one side</u> of the equation.**

So, for example, this equation is NOT solved for x: $2x + 6 = x$ because we have x's on both sides of the equation, and also because the x on the left side of the equation is not by itself.

On the other hand, this equation IS solved for p: $p = \frac{nrt}{v}$ because the p is by itself on one side of the equation. The fact that the other side of the equation still contains a bunch of other variables instead of just a single number is irrelevant.

<u>Now you try!</u> Which of the following equations have been solved for x, and which have not? <u>Circle each one that has been solved</u> for x, and <u>cross out any that have not yet been solved</u> for x and explain WHY they have not been solved for x.

1. $x = 2$
2. $-3 = x$
3. $x = 2y + 7$
4. $x = 3 + x^2 - 4$
5. $x = \frac{pqr}{s}$
6. $x < 9$
7. $8 \geq x$
8. $x = \frac{8}{5}$
9. $x = \sqrt{5}$

<u>How do we solve an equation for a specific variable?</u>
Just like with simplifying or rewriting expression, the process of solving an equation for a specific variable is really just a process of <u>replacing an equation with a sequence of different equations, all of which are equivalent to one another</u>.

How can we tell if two equations are equivalent? **Two equations (or inequalities, or systems of equations) are equivalent if they have the same solution set.**

The solution set is the set of all different combinations of numbers that could be substituted in for the variables that would make the equation true.

So, for example:

- The two equations $x - 5 = 2$ and $x^2 + 49 = 14x$ are equivalent because they both have the same solution set $x = 7$. As it turns out, 7 is the only value that we can substitute in for x that will make the first or the second equation true. Even though these two equations don't look at all similar, it turns out they are actually equivalent because they have the same solution set.
- The two equations $x + y = z$ and $(x + y) + 5 = z + 5$ are equivalent because they both have the same solution set: any combination of values that we substitute in for x, y, z in the first equation will also work in the second equation, since adding 5 to each of two equal things, still results in two equal things (although the two *new* equal things will not be equal to the two *old* equal things—each of the *new* things will be five *more* than each of the *old* things).
- **To show that two <u>equations</u> are equivalent, we can NOT use the equals sign.** Since equations already contain an equals sign, adding a third equals sign between two equations would be confusing (it would look like we are comparing expressions, instead of the whole equation). Instead, we use the double-arrow, like this:
  $x - 5 = 2 \quad \leftrightarrow \quad x^2 + 49 = 14x$
  $x + y = z \quad \leftrightarrow \quad (x + y) + 5 = z + 5$

So, to rewrite an equation, we simply need to replace it with an equivalent equation. The most systematic way to do this is to use properties (or identities) to substitute the equation for an equivalent equation (which we can also do by replacing a sub-expression with an equivalent sub-expression, just like with expressions).

This is just like what we did with expressions, except with equations we do have to think carefully about where we put the equals sign.

Example: Put the equation $-3(x-2) = y+1$ into the form $y = mx + b$ (where m, b are any numbers), and identify the values of m and b.	
work:	**explanation:**
$-3(x-2) = y+1$ $\leftrightarrow\ -3(x+-2) = y+1$	First, just as with simplifying expressions, we rewrite all subtraction as adding a negative (using the identity $a - b = a + -b$ and $a = x, b = 2$).
$[(-3)(x+-2)] = y+1$ $\leftrightarrow\ [(-3)(x) + (-3)(-2)] = y+1$ $\leftrightarrow\ -3x + (-3)(-2) = y+1$	Then, just like with expressions, we will try to replace the equation with an equivalent one that does not have parentheses, in this case using the distributive property to rewrite the sub-expression that is to the left of the equals sign: we notice that $-3(x+-2)$ has the form $a(b+c)$. Using $a=-3,\ b=x,\ c=-2$ for $a(b+c) = ab + ac$ yields: $(-3)(x+-2) = (-3)(x) + (-3)(-2)$ Substituting this back into the original equation (we use square brackets to do this) yields the result to the left. Since the left side of the equation always has an implied parentheses around it anyway, it doesn't change the meaning of the equation if we write it without the square brackets. Similarly, $(-3)(x)$ and $-3x$ are just two different ways of writing -3 times x, so we don't need the parentheses here either.
$-3x + [(-3)(-2)] = y+1$ $\leftrightarrow\ -3x + [6] = y+1$ $\leftrightarrow\ -3x + 6 = y+1$	Multiplying $-3 \cdot -2$ allows us to replace it with 6—we do this substitution using square brackets. We note that we don't need the square brackets around the 6 since it is just a single number (and so the brackets are not grouping anything).
$(-3x+6) = (y+1\ )$ $\leftrightarrow (\ y+1) = (-3x+6)$	Now, we need an equivalent equation of the form $y = mx + b$ (where m, b are any numbers). So we start by creating an equivalent equation that has the y on the left side and the x on the right side by using the property $a = b\ \leftrightarrow\ \ b = a$: Using $a = -3x+6,\ b = y+1$ for $a = b\ \leftrightarrow\ \ b = a$ yields the result to the left.
$(y+1) = (-3x+6)$ $\leftrightarrow (y+1) + -1 = (-3x+6) + -1$	Now we almost have what we need, but we need an equivalent equation that doesn't have the $+1$ that is on the left side (because in the form $y = mx + b$, the y is by itself on the left side). We look at the possible properties that we could use, and we notice that $1 + -1 = 0$ (so if we had an equivalent equation with $y + (1 + -1)$ on the left side, this would simplify to just $y + 0$ on the left side, which would simplify to just y on the left side, which is what we want). We notice that we have the following property which tells us that adding anything we want to each side of an equation produces an equivalent equation: $a = b\ \leftrightarrow\ \ a + c = b + c$ So, using $a = y+1,\ b = -3x+6$ and choosing $c = -1$ for $a = b\ \leftrightarrow\ \ a + c = b + c$ yields the result to the left.
$(y+1) + -1 = (-3x+6) + -1$ $\leftrightarrow y + 1 + -1 = -3x + 6 + -1$	If we think of each term as a single unit, we can see that all terms on the left are being added, and all terms on the right are being added, so we can replace the expressions on both the left and the right side of the equation with equivalent expressions that do not have the parentheses.
$y + [1 + -1] = -3x + [6 + -1]$ $\leftrightarrow\ \ y + [0] = -3x + [5]$	Since all the terms on the left and all the terms on the right are being added, we can combine the terms on a given side in any order (careful—we can **not** combine terms from one side with terms on the other side, because there is an equals sign between them!). So we can add $1 + -1$ first on the left and $6 + -1$ first on the right.
$[y+0] = -3x+5$ $\leftrightarrow\ [\ y] = -3x+5$ $\leftrightarrow\ \ y = -3x+5$ This has the form $y = mx + b$, with $m = -3,\ b = 5$.	We almost have what we need. We just need to replace the equation with an equivalent one that does not have the 0 on the left side. But we have an identity that says that $x + 0 = x$. So if we let $y = x$, we can rewrite this equation one last time by replacing the expression $y + 0$ on the left side of the equation with the expression y instead.

Notice two important things from this example:

1) Every time **we replaced the equation with an <u>equivalent</u> expression <u>using an identity</u>**.
2) When using the identities we were careful to **<u>use parentheses every time we substituted anything in for a variable and every time we substituted one sub-expression in for another</u>** (we could then only rewrite the equation without the parentheses afterwards if we could <u>find an identity</u> that showed us how to write <u>an equivalent equation</u> without the parentheses).

Now you try! For each of the following equations, put them into the requested form:

10. $2x - 3y = 6$	Put into $y = mx + b$ form (where m, b are any numbers) and give the values of m, b:
11. $y - 2 = -1(x - 3)$	Put into $y = mx + b$ form (where m, b are any numbers) and give the values of m, b:
12. $y = 5$	Put into $y = mx + b$ form (where m, b are any numbers) and give the values of m, b:

13. $2x - x^2 = 3 - x + 2x^2$	Put into $ax^2 + bx + c = 0$ form (where a, b, c are any numbers) and give the values of a, b, c:
14. $x^2 = 5 - x$	Put into $ax^2 + bx + c = 0$ form (where a, b, c are any numbers) and give the values of a, b, c:
15. $x^2 = 5$	Put into $ax^2 + bx + c = 0$ form (where a, b, c are any numbers) and give the values of a, b, c:

Use the properties on the first page of the project to solve each of the following equations, using these steps:

1) **If possible, first simplify the expressions on each side of the equation or inequality,**
2) **Then rewrite the equation as needed to get an equivalent equation with the indicated variable by itself on one side.**
3) **Then check your work by substituting the solution into the original equation.**

Original equation	**Solve for the variable indicated**	**Check your work**
Example: Solve for x: $\frac{2x}{2}+\frac{4}{3}=\frac{5}{6}$	**Minigoal 1: Find an equivalent equation with no fractions** The fractions are making this equation more complicated than we would like, so our first goal is to replace the equation with an equivalent one that doesn't have them. Fractions are just division, so one way to generate an equivalent equation is to multiply both sides by a multiple of all the denominators—6 is a multiple of 2, and 3, and 6. So if we multiply both sides of the equation by 6 using the identity $a=b \quad \leftrightarrow \quad a \cdot c = b \cdot c$, this should allow us to rewrite the equation without the fractions (notice that the **parentheses are critical here—if we leave them out, we change the meaning!**): Here $a=\frac{2x}{2}+\frac{4}{3},\ \ b=\frac{5}{6},\ \ c=6$: $\left(\frac{2x}{2}+\frac{4}{3}\right)=\left(\frac{5}{6}\right) \leftrightarrow 6\cdot\left(\frac{2x}{2}+\frac{4}{3}\right)=6\cdot\left(\frac{5}{6}\right)$ Now we do a series of substitutions, using other properties—here the square brackets show in each step what is being substituted in for what: $\left[6\cdot\left(\frac{2x}{2}+\frac{4}{3}\right)\right]=6\cdot\left(\frac{5}{6}\right) \leftrightarrow \left[6\cdot\left(\frac{2x}{2}\right)+6\cdot\left(\frac{4}{3}\right)\right]=6\cdot\left(\frac{5}{6}\right)$ $\left[6\cdot\left(\frac{2x}{2}\right)+6\cdot\left(\frac{4}{3}\right)\right]=6\cdot\left(\frac{5}{6}\right) \leftrightarrow 6\cdot\left(\frac{2x}{2}\right)+6\cdot\left(\frac{4}{3}\right)=6\cdot\left(\frac{5}{6}\right)$ $[6]\cdot\left(\frac{2x}{2}\right)+[6]\cdot\left(\frac{4}{3}\right)=[6]\cdot\left(\frac{5}{6}\right) \leftrightarrow \left[\frac{6}{1}\right]\cdot\left(\frac{2x}{2}\right)+\left[\frac{6}{1}\right]\cdot\left(\frac{4}{3}\right)=\left[\frac{6}{1}\right]\cdot\left(\frac{5}{6}\right)$ $\left[\frac{6}{1}\right]\cdot\left(\frac{2x}{2}\right)+\left[\frac{6}{1}\right]\cdot\left(\frac{4}{3}\right)=\left[\frac{6}{1}\right]\cdot\left(\frac{5}{6}\right) \leftrightarrow \frac{6}{1}\cdot\frac{2x}{2}+\frac{6}{1}\cdot\frac{4}{3}=\frac{6}{1}\cdot\frac{5}{6}$ $\left[\frac{6}{1}\cdot\frac{2x}{2}\right]+\left[\frac{6}{1}\cdot\frac{4}{3}\right]=\left[\frac{6}{1}\cdot\frac{5}{6}\right] \leftrightarrow \left[\frac{6\cdot 2x}{1\cdot 2}\right]+\left[\frac{6\cdot 4}{1\cdot 3}\right]=\left[\frac{6\cdot 5}{1\cdot 6}\right]$ $\left[\frac{6\cdot 2x}{1\cdot 2}\right]+\left[\frac{6\cdot 4}{1\cdot 3}\right]=\left[\frac{6\cdot 5}{1\cdot 6}\right] \leftrightarrow \frac{12x}{2}+\frac{24}{3}=\frac{30}{6}$ $\left[\frac{12x}{2}\right]+\frac{24}{3}=\frac{30}{6} \leftrightarrow \left[\frac{12}{2}x\right]+\frac{24}{3}=\frac{30}{6}$ $\left[\frac{12}{2}x\right]+\frac{24}{3}=\frac{30}{6} \leftrightarrow \frac{12}{2}x+\frac{24}{3}=\frac{30}{6}$ $\left[\frac{12}{2}\right]x+\left[\frac{24}{3}\right]=\left[\frac{30}{6}\right] \leftrightarrow [6]x+[8]=[5]$ $[6]x+[8]=[5] \leftrightarrow 6x+8=5$ So we have replaced $\frac{2x}{2}+\frac{4}{3}=\frac{5}{6}$ with the equivalent equation $6x+8=5$. **Minigoal 2: Find equivalent equation without the 8 on the left** Now that we no longer have fractions, we proceed to try to get x by itself on one side by thinking about what parts are "in the way". We need an equivalent equation without the 8 on the left side, and it is currently being added in the existing equation—this tells us that we can generate an equivalent equation by adding a -8 to each side of the equation using the identity $a=b \quad \leftrightarrow \quad a+c=b+c$, letting $a=6x+8, b=-3, c=-8$: $6x+8=5 \leftrightarrow (6x+8)+-8=(5)+-8$ $(6x+8)+-8=(5)+-8 \ \leftrightarrow 6x+(8+-8)=-3$ $6x+(8+-8)=-3 \leftrightarrow 6x+0=-3$ $6x+0=-3 \leftrightarrow 6x=-3$	$x=-\frac{1}{2}$ and $\frac{2x}{2}+\frac{4}{3}=\frac{5}{6}$ $\leftrightarrow \frac{2\left(-\frac{1}{2}\right)}{2}+\frac{4}{3}=\frac{5}{6}$ $\leftrightarrow \frac{\frac{2}{1}\left(-\frac{1}{2}\right)}{2}+\frac{4}{3}=\frac{5}{6}$ $\leftrightarrow \frac{-\frac{2\cdot 1}{1\cdot 2}}{2}+\frac{4}{3}=\frac{5}{6}$ $\leftrightarrow \frac{-\frac{2}{2}}{2}+\frac{4}{3}=\frac{5}{6}$ $\leftrightarrow \frac{-1}{2}+\frac{4}{3}=\frac{5}{6}$ $\leftrightarrow \frac{-1}{2}\cdot\frac{3}{3}+\frac{4}{3}\cdot\frac{2}{2}=\frac{5}{6}$ $\leftrightarrow \frac{-1\cdot 3}{2\cdot 3}+\frac{4\cdot 2}{3\cdot 2}=\frac{5}{6}$ $\leftrightarrow \frac{-3}{6}+\frac{8}{6}=\frac{5}{6}$ $\leftrightarrow \frac{-3+8}{6}=\frac{5}{6}$ TRUE! ✓

	So we have replaced $\frac{2x}{2}+\frac{4}{3}=\frac{5}{6}$ with the equivalent equation $6x=-3$. **Minigoal 3: Find equivalent equation without the 6 on the left** Now we need to replace this equation with one that does not have the 6 on the left side, and it is currently being multiplied in the equation—this tells us that we can generate an equivalent equation by dividing each side of the equation by 6 using the identity $a=b \leftrightarrow \frac{a}{c}=\frac{b}{c}$ $(whenever\ c\neq 0)$, letting $a=6x, b=-3, c=6$: $(6x)=(-3)\leftrightarrow \frac{(6x)}{6}=\frac{(-3)}{6}$ $\frac{(6x)}{6}=\frac{(-3)}{6}\leftrightarrow \frac{6x}{6}=\frac{-3}{6}$ $\left[\frac{6x}{6}\right]=\frac{-3}{6}\leftrightarrow \left[\frac{6}{6}x\right]=\frac{-3}{6}$ $\left[\frac{6}{6}x\right]=\frac{-3}{6}\leftrightarrow \left[\frac{6}{6}\right]x=\frac{-3}{6}$ $\left[\frac{6}{6}\right]x=\frac{-3}{6}\leftrightarrow [1]x=\frac{-3}{6}$ $[1]x=\frac{-3}{6}\leftrightarrow x=\frac{-1}{2}$ So we have replaced $\frac{2x}{2}+\frac{4}{3}=\frac{5}{6}$ with the equivalent equation $x=\frac{-1}{2}$ (or $x=-\frac{1}{2}$). **The only value of x that makes this equation true is $-\frac{1}{2}$, so that is also the only value of x that makes the original equation true.** That is why this is the **solution** to the **original equation**.	
16. $2(7-a)=6a-2$	Solve for a:	

17. $a = 6b + 2c$	Solve for b:	
18. $3x - 4y = 8$	Solve for y:	

19. $\frac{n+3}{2} = \frac{n+3}{3}$	Solve for n:	
20. $6p - 1 = 5 - 2(3p + 3)$	Solve for p:	