CYCLIC SUBGROUPS ARE QUASI-ISOMETRICALLY EMBEDDED IN THE THOMPSON-STEIN GROUPS

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Abstract. We give criteria for determining the approximate length of elements in any given cyclic subgroup of the Thompson-Stein groups $F(n_1, ..., n_k)$ in terms of the number of leaves in the minimal tree-pair diagram representative. This leads directly to the result that cyclic subgroups are quasi-isometrically embedded in the Thompson-Stein groups. This result also leads to the corollaries that $\mathbb{Z}^n$ is also quasi-isometrically embedded in the Thompson-Stein groups for all $n \in \mathbb{N}$ and that the Thompson-Stein groups have infinite dimensional asymptotic cone.

1. Introduction

In this paper, we continue an exploration of the metric of the Thompson-Stein groups that was begun in [16]; in [16], we gave sharp upper and lower bounds on the metric of $F(n_1, ..., n_k)$, showing that the lower bound is logarithmic and the upper bound is linear with respect to the number of leaves in the minimal tree-pair diagram representative of an element. In this paper, we explore how the number of leaves in the minimal tree-pair diagram grows as we take powers of an element. This leads to results about quasi-isometric embeddings of cyclic subgroups in $F(n_1, ..., n_k)$, but the long-term aim of this approach is much broader: to understand how the number of leaves in the minimal tree-pair diagram grows as we take general products of elements of $F(n_1, ..., n_k)$. Our aim in future will be to extend the results of this paper to more general classes of products so that we may better understand how to calculate the metric of all elements in $F(n_1, ..., n_k)$.

Burillo showed in [7] that $\mathbb{Z}$ and $\mathbb{Z}^n$ are quasi-isometrically embedded in F, and since the asymptotic cone of $\mathbb{Z}^n$ is infinite dimensional, this led directly to his result that F is the first known example of finitely-presented group with infinite-dimensional asymptotic cone. We extend these results to the Thompson-Stein groups here.

Melanie Stein was one of the first to study the Thompson-Stein groups in depth. In [14], Stein explored the homological and simplicity properties of $F(n_1, ..., n_k)$; she showed that they are of type FP_∞ and finitely presented, and gave a technique for computing infinite and finite presentations. In [16] we developed the theory of tree-pair diagram representation for elements of $F(n_1, ..., n_k)$, gave a unique normal form, and calculated sharp bounds on the metric in terms of the number of leaves in the minimal tree-pair diagram representative using Stein's presentations. These

1991 *Mathematics Subject Classification.* 20F65.

Key words and phrases. Thompson's group,piecewise-linear homeomorphisms,Stein groups, Higman groups, quasi-isometrically embedded subgroups, cyclic subgroups.

The author acknowledges support from the CUNY Scholar Incentive Award, and would like to thank the Technische Universität Berlin for its hospitality during the writing of this paper.

results were used to prove that the inclusion embedding of F into $F(n_1, ..., n_k)$ is distorted in [17]. Now we use the results of [16] to show that all cyclic subgroups, or copies of $\mathbb{Z}$, are quasi-isometrically embedded in $F(n_1, ..., n_k)$.

Definition 1.1 (Thompson-Stein group $F(n_1, ..., n_k)$)**.** *The Thompson-Stein group $F(n_1, ..., n_k)$, where $n_1, ..., n_k \in \{2, 3, 4, ...\}$ and $k \in \mathbb{N}$, is the group of piecewise-linear orientation-preserving homeomorphisms of the closed unit interval with finitely-many breakpoints in the ring $\mathbb{Z}[\frac{1}{n_1 \cdots n_k}]$ and slopes in the multiplicative group $\langle n_1, ..., n_k \rangle$ in each linear piece. Here $F = F(2)$.*

We could equivalently say that every element of $F(n_1, ..., n_k)$ is a continuous piecewise-linear map with the fixed endpoints $(0, 0)$ and $(1, 1)$ and finitely-many breakpoints in the ring $\mathbb{Z}[\frac{1}{n_1 \cdots n_k}]$ (see Figure 1 for a sample group element).

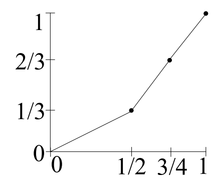


FIGURE 1. An element of $F(2, 3)$.

The Thompson-Stein groups can be viewed as a generalization of Thompson's group F, which Robert Thompson introduced in the early 1960s (see [15]). There are actually three groups in the literature that are frequently referred to as Thompson's groups, $F \subset T \subset V$; Cannon, Floyd and Parry provide a good introduction to these groups in [8]. Thompson's group F and the Thompson-Stein group $F(n_1, ..., n_k)$ (see [14]) are groups of piecewise-linear homeomorphisms of the real line. Higman, Brown, Geoghegan, Brin, Squier, Guzmán, Bieri and Strebel have all explored general classes in this family of groups, each of which can be considered to be a generalization of the Thompson groups (see [13], [5], [4], [2], [3], and [1] for details). Much of the introductory material in this paper is summarized from [16]; more detail can be found there.

The major results of this article hold for all groups of the form $F(n_1, ..., n_k)$ which satisfy the condition $n_1 - 1 | n_j - 1$ for all $j \in \{1, ..., k\}$. Groups which do not satisfy this criteria have a different group presentation.

2. Representing elements using tree-pair diagrams

Metric properties of the Thompson and Thompson-Stein groups depend on the representation of elements of these groups by tree-pair diagrams. An $(n_1, ..., n_k)$–ary tree is a tree which contains only carets whose number of edges are in $\{n_1, ..., n_k\}$. For example, Figure 2 is a $(2, 3, 5)$–ary tree. If each vertex in a tree represents

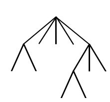


FIGURE 2. A $(2, 3, 5)$–ary tree.

a subinterval of $[0, 1]$, then every element of $F(n_1, ..., n_k)$ can be represented by an $(n_1,...,n_k)$–ary tree-pair diagram and vice versa. An $(n_1,...,n_k)$–ary tree represents subdivision of $[0,1]$ using the following recursive process, which assigns a subinterval to each leaf in the tree: the root vertex represents $[0,1]$; for a given n–ary caret in the tree with parent vertex representing $[a,b]$, the n child vertices represent

$$\left[a, a+\frac{1}{n}\right], \left[a+\frac{1}{n}, a+\frac{2}{n}\right], ..., \left[b-\frac{1}{n}, b\right]$$

respectively. So the subintervals for leaves of the tree in Figure 2 are:

$$\left\{\left[0,\frac{1}{10}\right],\left[\frac{1}{10},\frac{1}{5}\right],\left[\frac{1}{5},\frac{2}{5}\right],\left[\frac{2}{5},\frac{3}{5}\right],\left[\frac{3}{5},\frac{4}{5}\right],\left[\frac{4}{5},\frac{5}{6}\right],\left[\frac{5}{6},\frac{13}{15}\right],\left[\frac{13}{15},\frac{14}{15}\right],\left[\frac{14}{15},1\right]\right\}$$

Two $(n_1,...,n_k)$–ary trees are equivalent iff they represent the same subdivision of $[0,1]$.

The leaves in a tree are numbered beginning with zero, in increasing order from left to right, based on the order of their corresponding subintervals in $[0,1]$. An $(n_1,...,n_k)$–ary tree-pair diagram is then an ordered pair of $(n_1,...,n_k)$–ary trees with the same number of leaves. So the element of $F(n_1,...,n_k)$ represented by a given tree-pair diagram is the map which takes the subinterval represented by the ith leaf in the domain tree to the subinterval represented by the ith leaf in the range tree. Because every element of $F(n_1,...,n_k)$ is a piecewise-linear map with fixed endpoints, it can be determined solely by the length of ordered subintervals in the domain and range. For example, the element given in Figure 3 is just the map: $\left\{\left[0,\frac{1}{2}\right],\left[\frac{1}{2},\frac{3}{4}\right],\left[\frac{3}{4},1\right]\right\} \to \left\{\left[0,\frac{1}{3}\right],\left[\frac{1}{3},\frac{2}{3}\right],\left[\frac{2}{3},1\right]\right\}$, which is the same as the element of $F(n_1,...,n_k)$ represented by the map given in Figure 1.

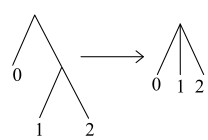


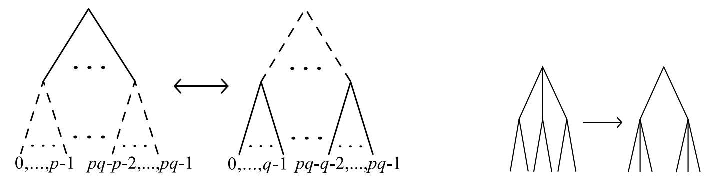
FIGURE 3. The element of $F(2,3)$ given by the homeomorphism in Figure 1.

Theorem 2.1 (Equivalent trees [16]). *Two trees are equivalent iff one tree can be obtained from the other through a finite sequence of subtree substitutions of the type given in Figure 4(a) (for any $p,q \in \{n_1,...,n_k\}$ such that $p \neq q$).*

For example, when $n_1 = 2$ and $n_2 = 3$, for $k = 2$, the only substitution of this type is given in Figure 4(b).

Notation 2.1 ($L(T)$, $L(T_-,T_+)$, $L(x)$). *The notation $L(T)$, $L(T_-,T_+)$, and $L(x)$ denotes the number of leaves in the tree T, in either tree of the tree-pair diagram (T_-,T_+), and in either tree of the minimal tree-pair diagram representative for x respectively.*

2.1. **Tree-pair diagram composition.** To compute xy for $x,y \in F(n_1,...,n_k)$, $x = (T_-,T_+)$, $y = (S_-,S_+)$, we must make S_+ identical to T_-. This can be accomplished by adding carets to T_- and S_+ (and by extension to the leaves with the same index numbers in T_+ and S_- respectively) until the valence of any leaves with the same index number in both T_- and S_+ is the same. If we let $T_-^*, T_+^*, S_-^* S_+^*$ denote



(a) Dotted line carets are p–ary and solid line carets are q–ary, where where $p, q \in \{n_1, ..., n_k\}, p \neq q$.

(b) Equivalent subtree substitution for $F(2,3)$.

FIGURE 4. Subtree substitutions of this form can be used to turn a tree into any equivalent tree.

T_-, T_+, S_-, S_+, respectively, after this addition of carets, then xy is represented by the (possibly non-minimal) diagram (S_-^*, T_+^*).

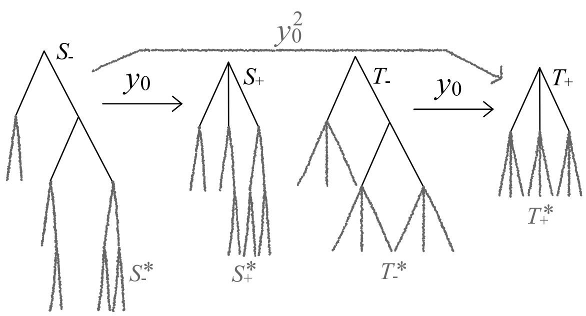


FIGURE 5. Composition of two elements of $F(2,3)$. The black carets make up the original tree-pair diagrams for each element, and grey hatched carets represent carets added so that composition can take place.

This process of composition always terminates.

Notation 2.2 $\big(((Ty)_-, (Ty)_+)\big)$**.** *When computing the product xy where $x = (T_-, T_+)$, the notation $\big((Ty)_-, (Ty)_+\big)$ denotes the tree-pair diagram which results from the composition of $x \circ y$, before it has been reduced by removing any exposed caret pairs.*

2.2. Equivalence of trees and tree-pair diagrams.

Corollary 2.1 (Wladis, [16])**.** *An $(n_1, ..., n_k)$–ary tree T can be written as an equivalent tree S with m–ary root caret if and only if S can be transformed into T through a finite sequence of subtree substitutions of the type given in Figure 4(a).*

Theorem 2.2 (Wladis, [16])**.** *Any two equivalent $(n_1, ..., n_k)$–ary tree-pair diagrams can be transformed into one another by a finite sequence consisting solely of the following two types of actions:*

(1) *addition or cancelation of exposed caret pairs*
(2) *subtree substitutions of the type given in Figure 4(a)*

3. The Metric in $F(n_1, ..., n_k)$

3.1. **Standard Finite Presentation.** In [14] Stein gave a method for finding the finite presentations for the groups $F(n_1, ..., n_k)$; in [16] the exact finite presentations were given explicitly.

Theorem 3.1 (Finite Presentation of Thompson's group $F(n_1, ..., n_k)$, [16])**.** *Thompson's group $F(n_1, ..., n_k)$, where $n_1 - 1 | n_i - 1$ for all $i \in \{1, ..., k\}$ has the following finite presentation:*
The generators of this presentation are:

$$(y_i)_0, ..., (y_i)_{n_i-1}, (z_j)_0, ..., (z_j)_{n_i-1}$$

where $i \in \{2, ..., k\}$, $j \in \{1, ..., k\}$. Tree-pair diagram representatives for these generators can be seen in Figure 6.

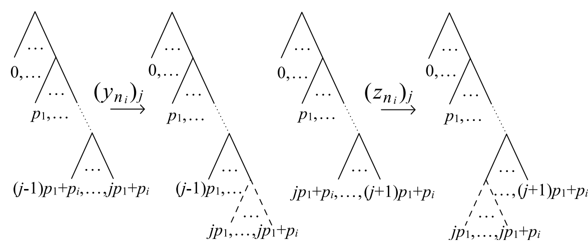


Figure 6. The generators of $F(n_1, ..., n_k)$, where $n_i = p_i + 1$ for all $i \in \{1, ..., k\}$ and where solid lines indicate n_1–ary carets and dashed lines indicate n_i–ary carets.

The relations of this presentation are:

(1) *For $(\gamma_m)_j$ a generator in the set $\{(y_m)_j, (z_m)_j | m \in \{1, ..., k\}\}$, $j = 0, ..., n_m - 1$*
  (a) $(z_l)_i^{-1}(\gamma_m)_j(z_l)_i = (z_l)_0^{-1}(\gamma_m)_j(z_l)_0$
  (b) $(z_l)_i^{-1}(z_l)_0^{-1}(\gamma_m)_j(z_l)_0(z_l)_i = (z_l)_0^{-2}(\gamma_m)_j(z_l)_0^2$
  (c) $(z_l)_{n_l-1}^{-1}(z_l)_0^{-2}(\gamma_m)_1(z_l)_0^2(z_l)_{n_l-1} = (z_l)_0^{-3}(\gamma_m)_j(z_l)_0^3$
  for all $l \in \{1, ..., k\}$ whenever $i < j$.

(2) *For all $i, j \in \{1, ..., k\}$, (where we use the convention that $(y_1)_i$ is the identity)*
  (a) $(y_i)_0(y_j)_1(z_j)_0(z_j)_{n_j} \cdots (z_j)_{(n_i-2)n_j} = (y_j)_0(y_i)_1(z_i)_0(z_i)_{n_i} \cdots (z_i)_{(n_j-2)n_i}$
  (b) $(y_i)_1(y_j)_2(z_j)_1(z_j)_{1+n_j} \cdots (z_j)_{1+(n_i-2)n_j} = (y_j)_1(y_i)_2(z_i)_1(z_i)_{1+n_i} \cdots (z_i)_{1+(n_j-2)n_i}$

(3) *For all $i, j \in \{1, ..., k\}$,*
  (a) $(z_i)_0(z_j)_0(z_j)_{n_j} \cdots (z_j)_{(n_i-1)n_j} = (z_j)_0(z_i)_0(z_i)_{n_i} \cdots (z_i)_{(n_j-1)n_i}$
  (b) $(z_i)_1(z_j)_1(z_j)_{1+n_j} \cdots (z_j)_{1+(n_i-1)n_j} = (z_j)_1(z_i)_1(z_i)_{1+n_i} \cdots (z_i)_{1+(n_j-1)n_i}$

3.2. **The Metric.** The metric in $F(n_1, ..., n_k)$ in this paper is always with respect to the standard finite generating set given in Section 3.1.

Theorem 3.2 (Wladis [16])**.** *For any element $x \in F(n_1, ..., n_k)$,there exist constants $b, c \in \mathbb{N}$ such that*

$$b \log L(x) \leq |x|_X \leq cL(x)$$

where we recall that $L(x)$ denotes the number of leaves in the minimal tree-pair diagram representative of w.

4. Cyclic Subgroups are Quasi-isometrically Embedded in $F(n_1, ..., n_k)$

Definition 4.1 (quasi-isometric embedding)**.** *We say that length functions $f : X \to \mathbb{R}^*, g : Y \to \mathbb{R}^*$ are quasi-isometric if there exists an embedding $\gamma : X \to Y$ and fixed $c_1, c_2 > 0$ such that for all $x \in X$:*

$$c_1 f(x) \leq g\big(\gamma(x)\big) \leq c_2 f(x)$$

When the length functions are obvious from the context, we say simply that that the sets X and Y are quasi-isometric.

We say that a subgroup S of a group G is quasi-isometrically embedded if the length function of S and the induced length function on $S \subset G$ coming from the inclusion embedding are quasi-isometric. The property of the existence of a quasi-isometric embedding between two groups is a group invariant.

The property of the existence of a quasi-isometric embedding between two sets is also transitive; that is, if $\sim$ denotes the property of being quasi-isometric, then if $X \sim Y$ and $Y \sim Z$, then $X \sim Z$.

We will use the notation $X \sim Y$ throughout this paper to indicate that X is quasi-isometric to Y.

The aim of this section is to show that all copies of $\mathbb{Z}$ in $F(n_1, ..., n_k)$ are quasi-isometrically embedded. Specifically we will show that all $x \in F(n_1, ..., n_k)$ break down into two cases (for $n \in \mathbb{N}$):

(1) $|x^n|_{F(n_1,...,n_k)}$ is quasi-isometric to $L(x^n)$.
(2) $|x^n|_{F(n_1,...,n_k)}$ is quasi-isometric to $\log(L(x^n))$.

We will give specific criteria which will allow us to determine whether any given element of $F(n_1, ..., n_k)$ is of type 1 or type 2 above, and this will immediately lead to a proof of the following theorem.

Theorem 4.1. *For all $x \in F(n_1, ..., n_k)$, where $\langle x \rangle$ represents the cyclic subgroup generated by x, $|x^n|_{F(n_1,...,n_k)}$ is quasi-isometric to $n = |x^n|_{\langle x \rangle}$.*

First we will show that if $L(x^n)$ grows exponentially with respect to n, then $|x^n|_{\langle x \rangle} \sim \log\big(L(x^n)\big) \sim |x^n|_{F(n_1,...,n_k)}$, and if $L(x^n)$ grows linearly with respect to n, then $|x^n|_{\langle x \rangle} \sim L(x^n) \sim |x^n|_{F(n_1,...,n_k)}$. Then the only task that will remain in order to proof Theorem 4.1 will be to show that any subgroup $\langle x \rangle \in F(n_1, ..., n_k)$ will fall into one of the following two subsets:

(1) $|x^n|_{\langle x \rangle}$ is quasi-isometric to $L(x^n)$.
(2) $|x^n|_{\langle x \rangle}$ is quasi-isometric to $\log(L(x^n))$.

Remark 4.1. *If $|x^n|_{\langle x \rangle} \leq d \log\big(L(x^n)\big)$ for fixed $d \in \mathbb{R}^+$ for all n, then*

$$|x^n|_{\langle x \rangle} \sim |x^n|_{F(n_1,...,n_k)} \sim \log\big(L(x^n)\big)$$

Proof. By Theorem 3.2 and the hypothesis, there exists $c \in \mathbb{R}^+$ such that for all n,

$$c \log\big(L(x^n)\big) \leq |x^n|_{F(n_1,...,n_k)} \leq |x^n|_{\langle x \rangle} \leq d \log\big(L(x^n)\big)$$

□

Definition 4.2 (leaf sets)**.** *For a given element $x = (T_-, T_+) \in F(n_1, ..., n_k)$, we let T_-^* and T_+^* denote the minimal trees that can be obtained from T_- and T_+ respectively by adding carets until $T_-^* \equiv T_+^*$. Then the negative leaf set of x (or the leaf set of T_-) is the set of leaf index numbers*

$$\{i | \text{carets must be added to the leaf } l_i \in T_- \text{ to obtain } T_-^*\}$$

We can similarly define the positive leaf set of x (or the leaf set of T_+).

Definition 4.3 (depth, $D(v)$, $D(T)$, $D(x)$, and level)**.** *The depth of a vertex v within a tree is the distance from the root vertex to v. The depth of a tree T is the maximum distance from the root vertex to any leaf vertex, and the depth of an element x is the maximal depth of the two trees in its minimal tree-pair diagram representative. We use $D(v)$, $D(T)$ and $D(x)$ to denote these depths, respectively. A level is the subgraph of carets in a tree which are the same distance from the root vertex.*

Lemma 4.1. *If $|x^n|_{\langle x\rangle} \geq dL(x^n)$ for fixed $d \in \mathbb{R}^+$ for all n, then*

$$|x^n|_{\langle x\rangle} \sim |x^n|_{F(n_1,\ldots,n_k)} \sim L(x^n)$$

Proof. It is clear that $|x^n|_{\langle x\rangle} \sim L(x^n)$ because putting Theorem 3.2 together with the hypothesis we have

$$dL(x^n) \leq |x^n|_{\langle x\rangle} \leq cL(x^n)$$

for some $c \in \mathbb{R}^+$ for all n.

Now in order to establish that $|x^n|_{F(n_1,\ldots,n_k)} \sim L(x^n)$ in this case, we prove that when $D(x) \sim L(x)$, $|x|_{F(n_1,\ldots,n_k)} \sim L(x)$. If $D(x)$ is quasi-isometric to $L(x)$, then there exists fixed $c_1 > 0$ such that $\frac{1}{c_1}L(x) \leq D(x) \leq c_1L(x)$. From the proof of Remark 5.2 in [16] we have that there exists $A \in \mathbb{N}$ such that $|x|_{F(n_1,\ldots,n_k)} \geq \frac{1}{A}D(x)$ for all $x \in F(n_1,...,n_k)$, and from Theorem 3.2 we have $|x|_{F(n_1,\ldots,n_k)} \leq CL(x)$ for a fixed $C \geq 1$ for all x. Putting this together yields $\frac{1}{Ac_1}L(x) \leq |x|_{F(n_1,\ldots,n_k)} \leq CL(x)$, which yields $\frac{1}{C_1}L(x) \leq |x|_{F(n_1,\ldots,n_k)} \leq C_1L(x)$ if we define $C_1 = \max\{Ac_1, C\}$.

Now to finish our proof, we need only prove that whenever $dL(x^n) \leq |x^n|_{\langle x\rangle}$ for some $d \in \mathbb{R}^+$ for all n, $L(x^n) \sim D(x^n)$. In this case, we already have $L(x^n) \sim |x^n|_{\langle x\rangle}$, so we only need to show that $D(x^n) \sim |x^n|_{\langle x\rangle}$. First we prove the lower bound: there exists $c_1 \in \mathbb{R}^+$ such that $|x^n|_{\langle x\rangle} \geq c_1D(x^n)$. It is clear that for all $x \in F(n_1,...,n_k)$, $L(x) \geq D(x)+1$. To see this we need only note that every level in a tree-pair diagram must have at least one caret in it; if we build a tree from the empty tree, the first caret we add will add at least two leaves to the tree, and every caret after that will add at least one leaf to the tree. So we have

$$dD(x^n) \leq dL(x^n) \leq |x^n|_{\langle x\rangle}$$

and we need only let $c_1 = d$.

Now we prove the upper bound; we now prove that for all $x \in F(n_1,...,n_k)$, there exists $c_2 \in \mathbb{R}^+$ such that for all n, $|x^n|_{\langle x\rangle} \leq c_2D(x^n)$. We consider the leaf with the smallest leaf index number in the leaf set of x and we denote this leaf by l_m. So by definition, when computing $x \cdot x$, the leaves $l_i \in T_-, T_+$ such that $i < m$ will have no carets added to them during composition. Without loss of generality, we suppose that l_m is in the leaf set of T_- (it may or may not also be in the leaf set of T_+). Let S denote the nonempty subtree that is added to $l_m \in T_-$ and let T denote the (possibly empty) subtree that is added to $l_m \in T_+$ during the composition $x \cdot x$. Our inductive hypothesis will be that x^n has a tree-pair diagram of the form given in Figure 7, that no carets on the leaf path of $l_m \in (Tx^n)_+$ in this diagram can be canceled, and that $D(x^n) \geq D(w_m) + (n-1)D(S)$.

It is clear that x^2 can be represented by the tree-pair diagram given in Figure 8. To see that none of the carets on the leaf path of $l_m \in (Tx^2)_+$ can be canceled, we need only note that the caret types in S were added to $l_m \in T_-$ during the

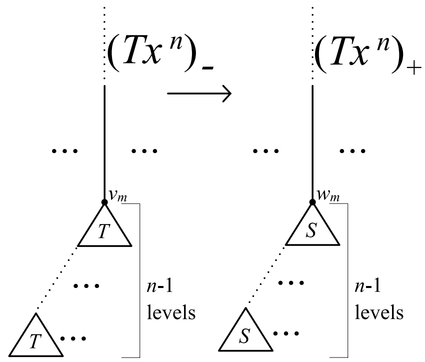


FIGURE 7. Tree-pair diagram representative for x^n.

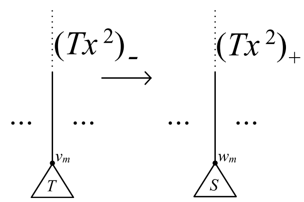


FIGURE 8. Tree-pair diagram representative for x^2.

composition $x \cdot x$ (and by extension to T_+) because they were present on the leaf path of $l_m \in T_+$ but not on the leaf path of $l_m \in T_-$. Likewise, any caret types in T were present on the leaf path of $l_m \in T_-$ but not on the leaf path of $l_m \in T_+$. So in order for any of the carets on the leftmost edge of S in $\big((Tx^2)_-(Tx^2)_+\big)$ to cancel, they would need to form an exposed caret pair with some carets present on the leaf path of $v_m \in (Tx^2)_-$, which is identical to the leaf path of $l_m \in T_-$, or with carets present in T, but the caret types present in T are identical to types already found on the leaf path of $l_m \in T_-$. So none of the carets on the leaf path of $l_m \in (Tx^2)_+$ can be canceled and therefore $D(x^2) \geq D(w_m) + D(S)$.

Now we suppose that our inductive hypothesis is true for $\big((Tx^{n-1})_-(Tx^{n-1})_+\big)$ and we consider the product $\big((Tx^{n-1})_-(Tx^{n-1})_+\big)(T_-, T_+)$, which is depicted in Figure 9. Since the subtrees S and T have no caret types in common, and since the leaf path of $v_m \in (Tx^{n-1})_-$ is identical to that of $l_m \in T_-$, we must add the subtree S to every descendant leaf of $v_m \in (Tx^{n-1})_-$, which includes the leaf $l_m \in (Tx^{n-1})_-$, so by extension we must add S to the leaf $l_m \in (Tx^{n-1})_+$. Likewise, we must add a subtree T to the leaf $l_m \in T_+$ and to $l_m \in T_-$ by extension; then, because $(Tx^{n-1})_-$ has a string of $n-2$–many T subtrees hanging off the vertex v_m, a string of $n-2$–many T trees must be added to the leaf $l_m \in T_+$ and by extension to the leaf $l_m \in T_-$, in addition to the single T tree already added. So the end result will be a tree-pair diagram of the form given in Figure 7. So all we need check is that none of the carets on the leaf path of $l_m \in (Tx^n)_+$ cancel. Again, because T and S have no caret types in common and because the carets in S were added because they were present in T_+ but not in T_-, and because the leaf paths of $v_m \in (Tx^n)_-$ and $w_m \in (Tx^n)_+$ are the same as the leaf paths of $l_m \in T_-$ and $l_m \in T_+$ respectively, we cannot have any exposed caret pairs containing $l_m \in \big((Tx^{n-1})_-(Tx^{n-1})_+\big)$. So

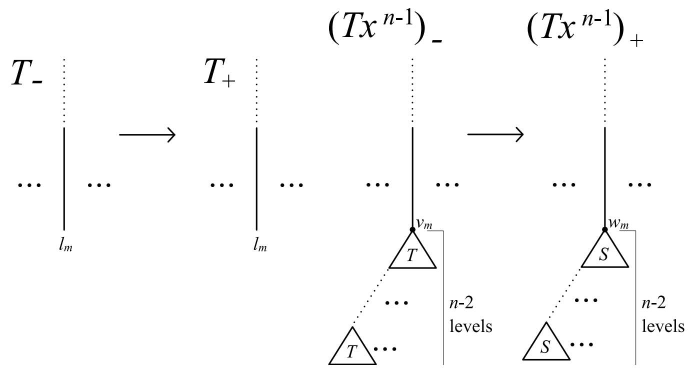


FIGURE 9. The product $x^{n-1} \cdot x$.

$D(x^n) \geq D(w_m) + (n-1)D(S)$. But since $|x^n|_{\langle x \rangle} = n$, it is clear that we have

$$|x^n|_{\langle x \rangle} \leq 2D(w_m) + 2(n-1)D(S) \leq 2D(x^n)$$

□

Now to complete the proof of Theorem 4.1 all that remains is to prove the following Lemma:

Theorem 4.2. *Any subgroup $\langle x \rangle \in F(n_1, ..., n_k)$ will fulfil one of the following conditions:*

1. *$L(x^n) \geq a^n$ for fixed $a \neq 1, a > 0$ for all n.*
2. *$L(x^n) \leq bn$ for fixed $b \in \mathbb{R}^+$ for all n.*

We note that condition 1 of Theorem 4.2 is simply a rewriting of the hypothesis of Remark 4.1 and that condition 2 is a rewriting of the hypothesis of Lemma 4.1. The proof of this theorem will be based on the idea that when certain "overlapping" mismatches of caret types occur in a tree-pair diagram, the number of leaves which must be added with each increase in the power will be a multiple of those added in the previous step, while when such and "overlapping mismatch" does not occur, the number of carets added with each increase in the power will remain constant. Before we can proceed to this proof, we need to first formalize this idea.

4.1. **Proof of Theorem 4.2.**

Definition 4.4 (Mismatches)**.** *A mismatch is a pair of carets of two distinct types in the trees T_- and T_+ with root vertices v_m and w_m respectively, such that v_m and w_m represent the same subinterval of the unit interval.*

The significance of mismatches is that, in order to make T_- equivalent to T_+ (as will be necessary when computing the product $x \cdot x$) any mismatch in T_- must have a caret of opposite type added to every leaf (see, for example the mismatches present in the composition given in Figure 5); if the index numbers of leaf descendants of the mismatch line up in the right way, then this will lead to an increasing number of carets being added to each tree with each new step in the product x^n, which will lead to exponential growth of $L(x^n)$.

1. **Irreducible mismatch:** *An irreducible mismatch is a mismatch for which no substitution of equivalent subtrees in (T_-, T_+) will allow the mismatch*

to be removed completely or for the mismatch to be moved to a lower level in (T_-, T_+).

In this paper, we will use the convention that all mismatches are irreducible, and so we will use the word mismatch to denote irreducible mismatches.

(2) **Leaf sets of mismatches:** *The negative leaf set of a mismatch in (T_-, T_+) is the subset of the negative leaf set of (T_-, T_+) containing all descendants of the half of the mismatched caret pair in T_-. We can likewise define the positive leaf set of a mismatch.*

(3) **Opposite mismatch types:** *We consider a caret $\wedge_i$ of type n_1 in the negative tree which is one half of a mismatch whose other half has type n_2 in the positive tree; we also consider a caret $\wedge_j$ of type m_2 in the positive tree which is one half of a mismatch whose other half has type m_1 in the negative tree. We say that the two "mismatch halves" $\wedge_i$ and $\wedge_j$ are of opposite mismatch type iff $m_1 \neq n_2$. For a concrete example of this, see Figure 10.*

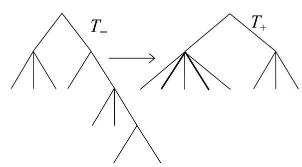


FIGURE 10. This tree-pair diagram contains two mismatches: $(\wedge_{0,1,2}, \wedge_{0,1,2,3,4})$ and $(\wedge_{3,4,5,6,7}, \wedge_{5,6,7})$ (where the index numbers given to the carets here are the leaf numbers of their descendants). The mismatch halves $\wedge_{0,1,2} \in T_-$ and $\wedge_{5,6,7} \in T_+$ are of opposite type, since $\tau(\wedge_{0,1,2,3,4} \in T_+) = 5 \neq \tau(\wedge_{3,4,5,6,7} \in T_-) = 2$. However, $\wedge_{0,1,2,3,4} \in T_+$ and $\wedge_{3,4,5,6,7} \in T_-$ are not of opposite type, since $\tau(\wedge_{0,1,2} \in T_-) = 3 = \tau(\wedge_{5,6,7} \in T_+)$.

(4) **Disjoint, overlapping mismatch halves:** *We say that two mismatch halves are disjoint if their leaf sets are disjoint. We say that two mismatch halves overlap if the intersection of their leaf sets is non-empty.*

(5) **Stable mismatches:** *We say that a mismatch half in y is stable if that mismatch half overlaps with another mismatch half in y^n for all n, or if that mismatch is disjoint from all other mismatches in y^n for all n. For example, each half of the mismatch which consists of the root caret in both trees of y_0 in Figure 5 is stable, because it will be overlapping for all y_0^n, and each half of the the mismatch which consists of the root caret in both trees of x in Figure 11 is stable, because it will be disjoint for all x^n. However, each half of the leftmost child of the rightmost child of the root in Figure 12 (i.e. the caret with leaf index numbers 3 and 4 in the negative tree and 1,2, and 3 in the positive tree) is not stable, because these mismatch halves overlap for x, but not for x^2.*

Our proof of Theorem 4.2 will then have the following structure: When there exists finite $N \in \mathbb{N}$ for a given $x \in F(n_1, ..., n_k)$ such that all the mismatches in $y \equiv x^N$ are stable, then:

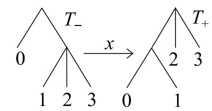


FIGURE 11. A tree-pair diagram with a stable disjoint mismatch.

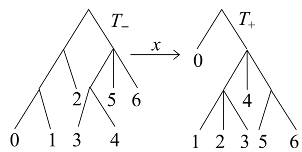


FIGURE 12. A tree-pair diagram which contains an unstable overlapping mismatch.

(1) If an overlapping mismatch exists in $y = x^N$, then $L(x^n) \geq a^n$ for fixed $a \neq 1, a > 0$ for all n.
(2) If an overlapping mismatch does not exist in $y = x^N$, then $L(x^n) \leq bn$ for fixed $b \in \mathbb{R}^+$ for all n.

It is clear that if this is true, Theorem 4.2 immediately follows. The proof of this theorem will have three parts: first we will consider the case when an overlapping mismatch does occur in y. Next we will show that for any $x \in F(n_1, ..., n_k)$, there exists fixed $N \in \mathbb{N}$ such that all the mismatches in $y \equiv x^N$ are stable, and finally we will explore the case in which no overlapping mismatch occurs in y.

Remark 4.2. *If a leaf in the leaf set is not a part of a mismatch, or if it belongs to a mismatch half that is stable and disjoint, then there exists $c \in \mathbb{R}^+$ such that the number of leaves contributed to x^n by that leaf is cn.*

Proof. It is clear that any leaves in the leaf set which are not a part of a mismatch will always contribute a fixed constant number of leaves in each step of the computation of the power x^n, because this is the same as the case $F(n)$. So we need only consider leaves that belong to stable disjoint mismatch halves. By definition, a mismatch half that does not overlap with any other mismatch half will have no carets added to it when computing powers - all added carets will be added to leaves that are not descended from the mismatch. So if a disjoint mismatch half is stable, the number of carets contributed to x^n by that mismatch half will be nc for some fixed constant c.

□

Lemma 4.2. *If l_m is the first leaf in the leaf set which belongs to an overlapping mismatch and it is stable, and if all leaves with indexes less than m that are in the leaf sets of mismatches are stable and disjoint, then $L(x^n) \geq a^n$ for fixed $a \neq 1, a > 0$ for all n.*

If this lemma holds, and if we can show that for all x there exists a fixed N such that all mismatches in $y = x^n$ are stable, then it immediately follows that anytime a stable overlapping mismatch exists in y, $L(y^n) \geq a^n$ for fixed $a \neq 1, a > 0$ for all n. So this lemma, coupled with Lemma 4.3 which we will prove below, will immediately lead to a proof of the first case of Theorem 4.2.

Proof. We show that whenever an overlapping mismatch exists in x, there exists fixed $a \neq 1, a > 0$ such that $L(x^n) \geq a^n$ for all n. We prove this by showing that if l_m is the first leaf in the leaf set which belongs to an overlapping mismatch, then x^n has the form given in Figure 13, where the number of leaves is clearly exponential with respect to n. Then we will need to prove that this tree-pair diagram cannot be significantly reduced.

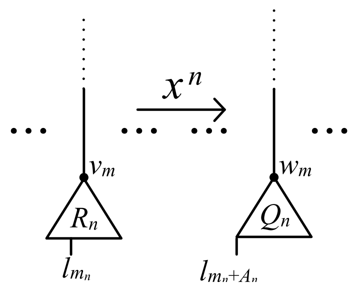


FIGURE 13. A tree-pair diagram for x^n. The leaf l_{m_n} in the domain tree denotes the leftmost leaf in the bottom level of R_n. Here $B = \min\{L(S), L(T)\}$, $b = L(S) - 1$, and $C = c_1 - c_2$; in this figure, $L(R_n), L(Q_n) \geq O(B^{n-1})$ and $A_n = O(b^{n-2})$. In addition, the subtrees Q_n and R_n contain $(n-1)$-many levels of S and T subtrees respectively, where the number of S and T subtrees respectively present in the ith level down in Q_n and R_n respectively is $O(B^{i-1})$.

We will prove that x^n has the form given in Figure 13 by induction. Again we let v_m and w_m represent the vertices in $(Tx^n)_-$ and $(Tx^n)_+$ respectively which represent the same subinterval of $[0,1]$ as $l_m \in T_-$ and $l_m \in T_+$ respectively. We let S denote the subtree that must be added to $l_m \in T_-$ in order to make T_- equivalent to T_+, and we let T denote the subtree that must be added to $l_m \in T_+$ in order to make T_+ equivalent to T_-. We let c_1 and c_2 denote the number of leaves added by leaves with index numbers less than m in the negative and positive leaf sets respectively; from Remark 4.2, we know that these leaves will be added to the left of $v_m \in (Tx^i)_-$ and $w_m \in (Tx^i)_+$ in each step in the product x^n. When $n = 2$, x^n has the form given in Figure 14; this clearly satisfies our inductive hypothesis.

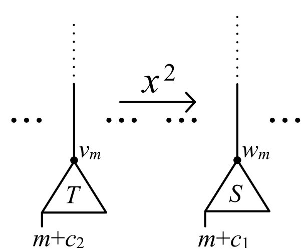


FIGURE 14. The tree-pair diagram for x^2.

Now we suppose that our inductive hypothesis holds, and we compute x^{n+1}. First we consider the product $x \cdot x^n$. When computing this product, we will have to add c_2-many leaves to the left of $l_{m_n+A_n} \in (Tx^n)_+$ and we will need to add a subtree T to every leaf in the subtree Q_n. So the new leaf index number of $l_{m_n+A_n} \in (Tx^n)_+$ in the resulting tree will be $m_n + A_n + c_2$. Now we must add all of these carets by extension to the leaves with the same index numbers in $(Tx^n)_-$ to obtain $(Tx^{n+1})_-$; so $l_{m_n+A_n} \in (Tx^{n+1})_-$ will be the leftmost leaf in a new level of T subtrees added to R_n, and therefore the index number of the leftmost leaf in the new lowest level in R_{n+1} will be $m_n + A_n + c_2$. The number of T subtrees added to the bottom level of R_n is then $\min\{L(Q_n), L(R_n) - (A_n + d_2)\}$ for some $d_2 \leq c_2$, so the new number of leaves in the bottommost level of R_{n+1} will be

$$L(T) \cdot \min\{L(Q_m), L(R_m) - (A_n + d_2)\}$$

Now we consider the product $x^n \cdot x$. When computing this product, we will have to add c_1-many leaves to the left of $l_{m_n} \in (Tx^n)_-$ and we will need to add a subtree S to every leaf in the subtree R_n. So the new leaf index number of $l_{m_n} \in (Tx^n)_-$ in the resulting tree will be $m_n + c_1$. Now we must add all of these carets by extension to the leaves with the same index numbers in $(Tx^n)_+$ to obtain $(Tx^{n+1})_+$; so a new level of S subtrees will be added to Q_n and the leftmost leaf in this level will have index number $m_n + A_n + c_1 + A_n \cdot b$, where $b = L(S) - 1$. The number of S subtrees added to the bottom level of Q_n is $\min\{L(Q_n), L(R_n) - (A_n)\}$, so the new number of leaves in the bottommost level of Q_{n+1} will be

$$L(S) \cdot \min\{L(Q_m), L(R_m) - (A_n)\}$$

Now we compute A_{n+1}:

$$A_{n+1} = (m_n + A_n + c_1 + A_n \cdot b) - (m_n + A_n + c_2) = bA_n + C$$

where we recall that $C = c_1 - c_2$. We can then use this to quickly prove the following inductive hypothesis:

$$A_n = O(b^{n-2})$$

Now we look at the number of leaves in the bottommost levels of R_{n+1} and Q_{n+1}. In R_{n+1} we have that the number of leaves in the bottommost level is:

$$\begin{aligned} &\geq \quad L(T) \cdot \min\{L(Q_n), L(R_n) - (A_n + d_2)\} \\ &= \quad L(T) \cdot \min\{O(B^{n-1}), O(B^{n-1}) - O(b^{n-2})\} \end{aligned}$$

Then there are two possibilities:

(1) If $B \geq b$, then $O(B^{n-1}) - O(b^{n-2}) = O(B^{n-1})$.
(2) If $b < B$, then $O(B^{n-1}) - O(b^{n-2}) = O(b^{n-2})$.

We recall that by definition, l_m is a stable overlapping mismatch. This means that the T subtrees must be added to leaves in R_n when computing x^{n+1} from $x \cdot x^n$, and therefore we must have $L(Q_n) - A_n > 0$ for all n. But since we have $O\big(L(Q_n) - A_n\big) = O(B^{n-1}) - O(b^{n-2})$, this implies that we must have $b < B$. So we have:

$$L(T) \cdot \min\{O(B^{n-1}), O(B^{n-1}) - O(b^{n-2})\} = L(T) \cdot O(B^{n-1}) \geq O(B^n)$$

Similarly, in Q_{n+1} we have that the number of leaves in the bottommost level is:

$$\geq L(S) \cdot \min\{L(Q_m), L(R_m) - A_n\} \geq O(B^n)$$

So it is clear that x^n has the form given in Figure 13, and it is clear that the number of leaves in this tree-pair diagram grows exponentially with respect to n, so now we need only show that this tree-pair diagram cannot be significantly reduced.

We consider the lowest level of T subtrees in $R_n \in (Tx^n)_-$ and the lowest level of S subtrees in $Q_n \in (Tx^n)_+$. Both of these levels have $O(B^{n-1})$ many leaves; these leaves in $(Tx^n)_-$ begin with leaf index number m_n, and in $(Tx^n)_+$ with leaf index number $m_n + A_n$. So the number of leaves in $\big((Tx^n)_-, (Tx^n)_+\big)$ which are in the bottommost level of T subtrees in $(Tx^n)_-$ and the bottommost level of S subtrees in $(Tx^n)_+$ must be at least

$$O(B^{n-1}) - O(A_n) = O(B^{n-1}) - O(b^{n-2}) = O(B^{n-1})$$

No we let l_i denote one of these leaves in $\big((Tx^n)_-, (Tx^n)_+\big)$; by definition, T and S have no caret types in common, so none of the carets on the leaf path of l_i which belong to T subtrees in $(Tx^n)_-$ or S subtrees in $(Tx^n)_+$ can cancel one another. The only way for a caret on the leaf path of l_i to cancel will be if subtree substitution of the type given in Figure 4(a) allows us to move carets present on the leaf path of $v_m \in (Tx^n)_-$ or $w_m \in (Tx^n)_+$ to the bottom level (we recall that the leaf path of $v_m \in (Tx^n)_-$ and $w_m \in (Tx^n)_+$ is identical to that of $l_m \in T_-$ and $l_m \in T_+$ respectively). But the number of carets on the leaf path of $v_m \in (Tx^n)_-$ and $w_m \in (Tx^n)_+$ is $D\big(v_m \in (Tx^n)_-\big)$ and $D\big(w_m \in (Tx^n)_+\big)$ respectively, and we have $D\big(v_m \in (Tx^n)_-\big), D\big(w_m \in (Tx^n)_+\big) \leq D(x)$, where $D(x)$ is a fixed constant. So the maximum number of leaves which can be canceled in this way will be $L(x)$, which is a fixed constant that is independent of n. Therefore, the number of leaves in the minimal tree-pair diagram for x^n in this case must be at least $O(B^{n-1})$. $\square$

Lemma 4.3. *For any $x \in F(n_1, ..., n_k)$, there exists fixed $N \in \mathbb{N}$ such that all the mismatches in $y \equiv x^N$ are stable.*

Proof. The basic idea behind this proof is simple: we show that if two unstable mismatch halves go from being disjoint to overlapping as we take powers, they either perpetually overlap or they move to the "opposite side" of one another (i.e. their left-right ordering is reversed), and we show that if two unstable mismatch halves go from being overlapping to disjoint as we take powers, they cannot overlap again.

We begin by letting m_0 be the lowest index number of the leftmost leaf of an unstable mismatch half in $x = (T_-, T_+)$; without loss of generality, we can suppose that this mismatch half is in T_-. We let m_i represent the index number of the leftmost leaf descendent of the vertex in $\big((Tx^{i+1})_-, (Tx^{i+1})_+\big)$ which represents the same subinterval of $[0,1]$ as $l_{m_0} \in T_-$. We let a_i and b_i represent the number of leaves which must be added to the left of l_{m_i} in $(Tx^i)_+$ and $(Tx^i)_-$, respectively, in order to compute x^{i+1}.

We can assume that whenever $a_n \leq b_n$, $a_{n+1} \leq b_{n+1}$ and that whenever $a_n \geq b_n$, $a_{n+1} \geq b_{n+1}$. To see that this is true, we need only claim that $a_n = O\big(c_1 a_1^{n-e_1} + \cdots + c_r a_r^{n-e_r} + d_1(n-p_1) + \cdots + d_r(n-p_r)\big)$ and $b_n = O\big(C_1 A_1^{n-E_1} + \cdots + C_s A_s^{n-E_s} + D_1(n-P_1) + \cdots + D_s(n-P_s)\big)$. It is clear that this is true because by Lemma 4.2 all leaves in the leaf set which do not belong to mismatches, or which belong to stable disjoint mismatches will add $O(cn)$ leaves to x^n for some fixed $c \in \mathbb{R}^+$, and because by Lemma 4.2, all mismatches which are stable and overlapping will add $O(a^n)$ leaves to x^n for some fixed $a > 0, a \neq 1$. So it is clear that there exists an $N \in \mathbb{N}$ such that for all x^n with $n \geq N$, $a_n \leq b_n$ or for all x^n with $n \geq N$ $a_n \geq b_n$. We then need only define $y = x^N$ and replace all x's in the remainder of this proof with y's.

For this proof we must consider two cases. We let l_{p_0} denote a leaf in the leaf set of the first mismatch half in T_+ with index number greater than or equal to m_0. We use similar notation for l_{p_0} as we defined for l_{m_0} above. Then there are two possibilities:

(1) The leaves l_{m_0} and l_{p_0} do not overlap in (T_-, T_+), so $m_0 < p_0$; when taking powers we eventually obtain $m_n = p_n$ for some n.
(2) The leaves l_{m_0} and l_{p_0} overlap in (T_-, T_+), so $m_0 = p_0$; when taking powers we eventually obtain $m_n < p_n$ for some n.

These are the only two ways in which a mismatch can be unstable, by definition.

We begin by considering case 1. This can only occur if the "distance" between l_{m_n} and l_{p_n} shrinks as n grows, i.e. if $p_n - m_n \leq p_{n-1} - m_{n-1}$ for some n. So we suppose that we have $m_i < p_i$ for all $i = 1, ..., n-1$, and we suppose that $p_n = m_n$. To construct the domain tree of x^{n+1}, we consider $x \cdot x^n$; we must add b_n leaves to the left of $l_{m_{n-1}}$ and a subtree T to $l_{m_{n-1}}$ in $(Tx^n)_+$ and by extension these leaves are added to the left of $l_{m_{n-1}}$ and the subtree to $l_{m_{n-1}}$ in $(Tx^n)_-$. So we will have $m_n = m_{n-1} + b_n$. Now we construct the range tree of x^{n+1} by considering the product $x^n \cdot x$; we must add a_n leaves to the left of $l_{m_{n-1}}$ and a subtree S to $l_{m_{n-1}}$ in $(Tx^n)_-$ and by extension these leaves are added to the left of $l_{m_{n-1}}$ and the subtree to $l_{m_{n-1}}$ in $(Tx^n)_+$. Since $m_{n-1} < p_{n-1}$, all of these leaves will be added to the left of $l_{p_{n-1}}$ in $(Tx^n)_+$. So we will have $p_n \geq p_{n-1} + a_n + L(S) - 1$. So we can only have $p_n = m_n$ whenever $b_n - \big(a_n + L(S) - 1\big) > 0$, which will only happen as long as $b_n > a_n$ and $b_n \geq L(S)$.

Now we compute x^{n+2} and consider the relationship between p_{n+1} and m_{n+1}. To construct the domain tree of x^{n+2}, we consider $x \cdot x^{n+1}$; we must add b_{n+1} leaves to the left of l_{m_n} and a subtree T to l_{m_n} in $(Tx^{n+1})_+$ and by extension these leaves are added to the left of l_{m_n} and the subtree to l_{m_n} in $(Tx^{n+1})_-$. So we will have $m_{n+1} = m_n + b_{n+1}$. Now we construct the range tree of x^{n+2} by considering the product $x^{n+1} \cdot x$; we must add a_{n+1} leaves to the left of l_{m_n} and a subtree S to l_{m_n} in $(Tx^{n+1})_-$ and by extension these leaves are added to the left of l_{m_n} and the subtree to l_{m_n} in $(Tx^{n+1})_+$. Since $m_n = p_n$, all of these leaves will be added to the left of l_{p_n} in $(Tx^{n+1})_+$, and S is added to l_{p_n}. So we will have $p_{n+1} \geq p_n + a_{n+1}$. Since we had $b_n > a_n$, and this implies that $b_{n+1} \geq a_{n+1}$, which yields $p_{n+1} \leq m_{n+1}$. So, once a pair of disjoint mismatch halves becomes overlapping, they either remain overlapping, or they "cross" one another so that the mismatch that was to the left is now to the right, and vice versa. Now we will show that when overlapping mismatches become disjoint, they never overlap with one another again.

We suppose that $m_0 = p_0$, and without loss of generality, we can suppose that $a_1 > b_1$. We consider the product x^2: to compute this product, we must add a_1-many leaves to the left of l_{m_0} and a subtree S to l_{m_0} in T_-, and by extension these leaves are added to the left of l_{m_0} and the subtree is added to l_{m_0} in T_+. Since $m_0 = p_0$, we have $p_1 = p_0 + a_1$. Similarly we compute $m_1 = m_0 + b_1$. So clearly $p_1 > m_1$.

Now we suppose that $p_i > m_i$ for all $i = 1, ..., n$; since $a_i > b_i \rightarrow a_{i+1} > b_{i+1}$, a simple induction argument will yield $a_i > b_i$ for all $i = 1, ..., n$. Now we compute x^{n+2} and consider the relationship between p_{n+1} and m_{n+1}. To construct the domain tree of x^{n+2}, we consider $x \cdot x^{n+1}$; we must add b_{n+1} leaves to the left of l_{m_n} and a subtree T to l_{m_n} in $(Tx^{n+1})_+$ and by extension these leaves are added to the left of l_{m_n} and the subtree to l_{m_n} in $(Tx^{n+1})_-$. So we will have $m_{n+1} = m_n + b_{n+1}$. Now we construct the range tree of x^{n+2} by considering the product $x^{n+1} \cdot x$; we must add a_{n+1} leaves to the left of l_{m_n} and a subtree S to l_{m_n} in $(Tx^{n+1})_-$ and by extension these leaves are added to the left of l_{m_n} and the subtree to l_{m_n} in $(Tx^{n+1})_+$. Since $m_n < p_n$, all of these leaves will be added to the left of l_{p_n} in $(Tx^{n+1})_+$, and S is added to l_{p_n}. So we will have $p_{n+1} \geq p_n + a_{n+1} + L(S) - 1$. Since we had $b_n > a_n$, and this implies that $b_{n+1} \geq a_{n+1}$, this yields $p_{n+1} \geq m_{n+1}$. So, once a pair of overlapping mismatch halves becomes disjoint, they never again overlap; rather, their relative positions to the left and right of one another are preserved. $\square$

Now we proceed to prove Theorem 4.2; the main result of this paper will then immediately follow.

Proof of Theorem 4.2. This proof has two cases:

(1) $L(x^n) \geq a^n$ for fixed $a \neq 1, a > 0$ for all n.
(2) $L(x^n) \leq bn$ for fixed $b \in \mathbb{R}^+$ for all n.

For the duration of this proof, we assume that all mismatches in x are stable; we can do this because, if they are not stable, we simply choose N such that all mismatches in $y \equiv x^N$ are stable, and then our proof will hold for y (and by extension, all our quasi-isometry results will hold for x as well).

We have already proven the fist case as it follows directly from the combination Lemma 4.2 and Lemma 4.3, so we now proceed to the second case; we will show that whenever no overlapping mismatches exist in x, there exists fixed $b \in \mathbb{R}^+$ such that $L(x^n) \leq bn$ for all n.

If no mismatches exist in x, it is clear that this is true, as this is identical to the situation in $F(n_i)$. So we consider what happens to a stable mismatch that does not overlap; we let l_m be a leaf in the leaf set of x which is in a mismatch such that all leaves in the leaf set with index less than m add only a fixed constant number of leaves to the tree-pair diagram in each step of the computation of x^n. Without loss of generality, we can assume that l_m is in the negative leaf set. Our goal will be to show that any leaf l_m which satisfies these conditions will contribute $(n-1)C$ leaves to the tree-pair diagram for x^n for some fixed $C \geq 0$. It is clear that if this is true, and that all mismatches in x are disjoint, then $L(x^n) \leq bn$ for some $b \in \mathbb{R}^+$.

We let $l_p \in T_+$ be the leftmost leaf in the leaf set of a mismatch in x that may or may not be distinct from the mismatch containing l_m, so that $p > m$. We let v_m and w_m represent the vertices in $(Tx^n)_-$ and $(Tx^n)_+$ respectively which represent the same subinterval of $[0,1]$ as $l_m \in T_-$ and $l_m \in T_+$, and likewise we let v_p and w_p represent the vertices in $(Tx^n)_-$ and $(Tx^n)_+$ respectively which represent the same subinterval of $[0,1]$ as $l_p \in T_-$ and $l_p \in T_+$.

We consider what happens when we take powers of x and we look specifically at the number of carets that must be added to the positive tree at each step. We show inductively that the number of carets added to the positive tree by l_m in the negative leaf set during the process of computing x^n will be $(n-1)L(S)$, where S is the subtree that must be added to l_m in T_- to make T_- equivalent to T_+. Our inductive hypothesis will be that there exist constants $c_1, c_2 \geq 0$ such that the leftmost leaf descended from $v_m \in (Tx^n)_-$ will have index number $m + (n-1)c_2$ and that the leftmost leaf descended from $w_m \in (Tx^n)_+$ will have index number $p + (n-1)\big(c_1 + L(S) - 1\big)$. Specifically, c_1 and c_2 will be the number of leaves added to T_- and to T_+ respectively to the left of l_m by leaves in the leaf set with index number less than m. From the proof of Remark 4.2, we also have that since $m < p$ and all the leaves in the leaf set with index number less than m add only a fixed number of leaves in each step, that we must have $c_2 < c_1$.

It is clear that our inductive hypothesis holds when $n = 1$. Now we suppose that it is true for n and we consider the product $x^n \cdot x$. From the proof of Remark 4.2, we know that since all the leaves in the leaf set with indexes below m add only a fixed number of leaves in each step, the number of leaves added to the left of the leftmost leaf descendent of v_m in the negative tree and w_m in the positive tree in each step will be the constant c_1 and c_2 respectively. So, when computing $x^n \cdot x$, we must add S to $l_{m+(n-1)c_2}$ and we must add c_1-many other leaves to the left of $l_{m+(n-1)c_2} \in (Tx^n)_-$, all of which come from leaves in the original leaf set of x with indices less than m; then by extension, we add S and c_1–many leaves to the left of $l_{p+(n-1)\left(c_1+L(S)-1\right)}$ in $(Tx^n)_+$, since we must have $m + (n-1)c_2 < p+(n-1)\left(c_1+L(S)-1\right)$. We already know from the proof of Remark 4.2 that nc_2 leaves will be added to the left of l_m in the negative tree when computing x^{n-1}, so it is clear that our inductive hypothesis holds for $n-1$. So when no overlapping mismatches exist in x, $L(x^n) \leq bn$ for fixed $b \in \mathbb{R}^+$ for all n. □

Remark 4.3. *$F(n_1, ..., n_k)^n$ is quasi-isometrically embedded in $F(n_1, ..., n_k)$.*

Proof. This is true for the same reasons that F^n is quasi-isometrically embedded in F; see [7], [9], [11], and [12] for details. The idea is that every copy of $F(n_1, ..., n_k)$ in the direct product can be mapped to a distinct product of subintervals of $[0, 1]$ in a specific way; for example, to embed $F \times F$ into F, one need only map the first factor to $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$ and the second factor to $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$. To adapt this for $F(n_1, ..., n_k)$, we can, for example, map the *ith* factor in the product $F(n_1, ..., n_k)^n$ to the subset of the unit square: $[\frac{i-1}{n_1}, \frac{i}{n_1}] \times [\frac{i-1}{n_1}, \frac{i}{n_1}]$. □

Corollary 4.1 (Corollary to Theorem 4.1)**.** *$\mathbb{Z}^n$ is quasi-isometrically embedded in $F(n_1, ..., n_k)$ for all $n \in \mathbb{N}$.*

Proof. This follows immediately from Theorem 4.1 and Remark 4.3. □

Corollary 4.2. *The asymptotic cone of $F(n_1, ..., n_k)$ is infinite dimensional.*

Proof. See [10] or [6]. □

References

[1] Robert Bieri and Ralph Strebel, *On groups of PL homeomorphisms of the real line*, preprint, Math. Sem. der Univ. Frankfurt, Frankfurt am Main, 1985.

[2] Matthew G. Brin and Fernando Guzmán, *Automorphisms of generalized Thompson groups*, J. Algebra **203** (1998), no. 1, 285–348. MR MR1620674 (99d:20056)

[3] Matthew G. Brin and Craig C. Squier, *Presentations, conjugacy, roots, and centralizers in groups of piecewise linear homeomorphisms of the real line*, Comm. Algebra **29** (2001), no. 10, 4557–4596. MR MR1855112 (2002h:57047)

[4] K.S. Brown, *Finiteness properties of groups*, J. Pure App. Algebra **44** (1987), 45–75.

[5] K.S. Brown and Geoghegan. R., *An infinite-dimensional torsion-free $FP1$ group*, Invent. Math. **77** (1984), 367–381.

[6] José Burillo, *Dimension and fundamental groups of asymptotic cones*, J. London Math. Soc. (2) **59** (1999), no. 2, 557–572. MR MR1709665 (2000i:20067)

[7] ———, *Quasi-isometrically embedded subgroups of Thompson's group F*, J. Algebra **212** (1999), no. 1, 65–78. MR MR1670622 (99m:20051)

[8] J.W. Cannon, W.J. Floyd, and W.R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. **42** (1996), 215–256.
[9] Sean Cleary and Jennifer Taback, *Geometric quasi-isometric embeddings into Thompson's group F*, New York J. Math. **9** (2003), 141–148 (electronic). MR MR2016187 (2004k:20083)
[10] Gromov, *Asymptotic invariants of infinite groups.*
[11] V. S. Guba and M. V. Sapir, *On subgroups of the R. Thompson group F and other diagram groups*, Mat. Sb. **190** (1999), no. 8, 3–60. MR MR1725439 (2001m:20045)
[12] Victor Guba and Mark Sapir, *Diagram groups*, Mem. Amer. Math. Soc. **130** (1997), no. 620.
[13] G. Higman, *Finitely presented infinite simple groups*, Notes on Pure Math. **8** (1974).
[14] Melanie Stein, *Groups of piecewise linear homeomorphisms*, Trans. Amer. Math. Soc. **332** (1992), no. 2, 477–514.
[15] R. J. Thompson and R. McKenzie, *An elementary construction of unsolvable word problems in group theory*, Word problems, Conference at University of California, Irvine, North Holland, 1969 (1973).
[16] Claire Wladis, *The Word Problem and the Metric for Generalizations of Thompson's Group F on more than One Integer*, Preprint at http://arxiv.org/abs/0811.3036.
[17] ———, *Thompson's Groups are Distorted in the Thompson-Stein Groups*, Preprint.

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